$\mathcal{H}_2$-optimal Model Order Reduction Using Projected Nonlinear Least Squares

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Mathworks

Mathematics of Reduced Order Models @ ICERM (21 February 2020)

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Paper: `https://arxiv.org/abs/1811.11962`
Code: `https://github.com/jeffrey-hokanson/MOR`

Given a Hilbert space $\mathcal{X}$ with inner product $\langle \cdot, \cdot \rangle$ set $\mathcal{M} \subset \mathcal{X}$, $y \in \mathcal{X}$

Original Problem

$$\widehat{x} = \underset{x \in \mathcal{M}}{\operatorname{argmin}} \, \|x - y\|^2 \tag{O}$$

Projected Problem

$$\widehat{x}^P = \underset{x \in \mathcal{M}}{\operatorname{argmin}} \, \|P[x - y]\|^2 \tag{P}$$

orthogonal projector $P : \mathcal{X} \to \mathcal{X}$

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Tangent cone $T_x\mathcal{M} := \{t : \exists \{\epsilon_k\}_{k=1}^{\infty} \geq 0,\ \{x_k\}_{k=1}^{\infty} \subset \mathcal{M},\ x_k \to x,\ \epsilon_k(x_k - x) \to t\}$

$=$"derivatives of paths in $\mathcal{M}$ passing through x"

Locally optimality & tangent cone definition from Hinze, Pinnau, Ulbrich, Ulbrich 2009

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Projection Lemma

[H. & Magruder]

If Range $P \supseteq T_x\mathcal{M}$ and x is a local optimizer of (P), then x is a local optimizer of (O)

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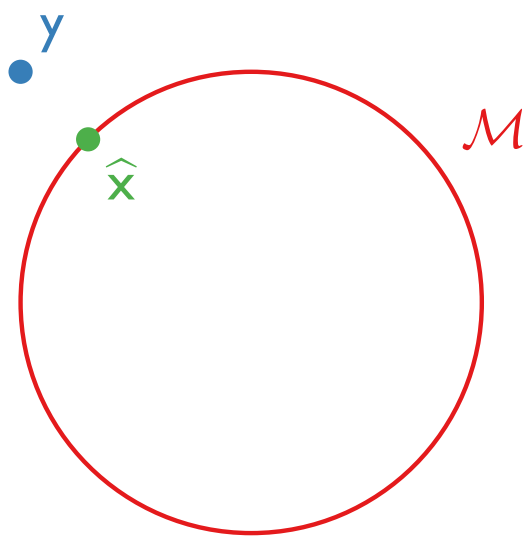

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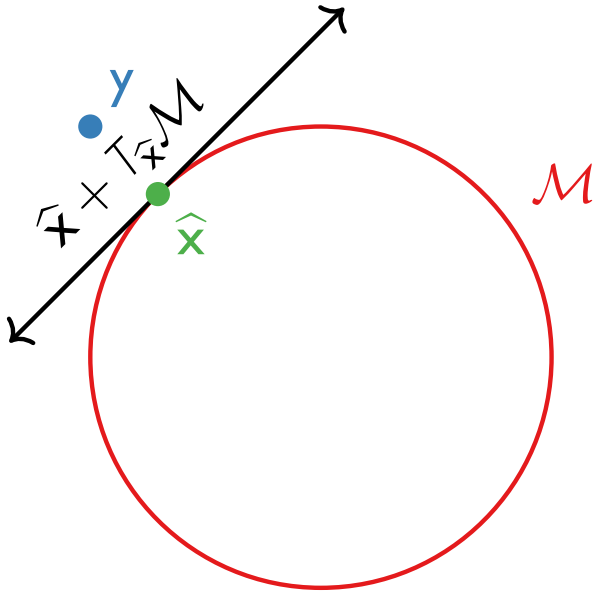

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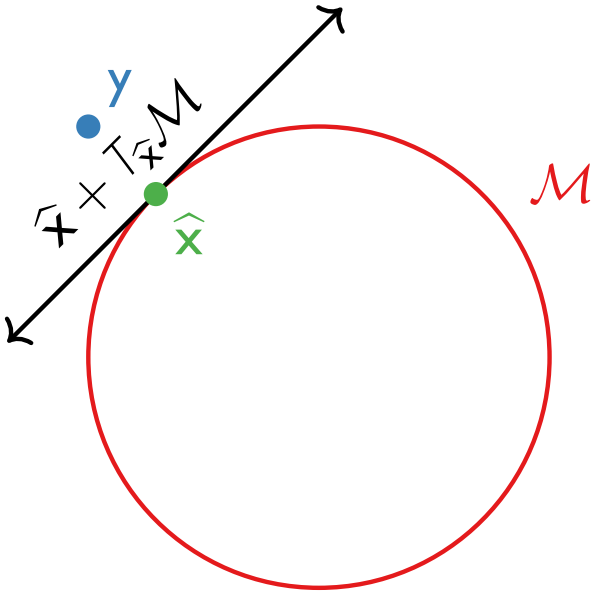

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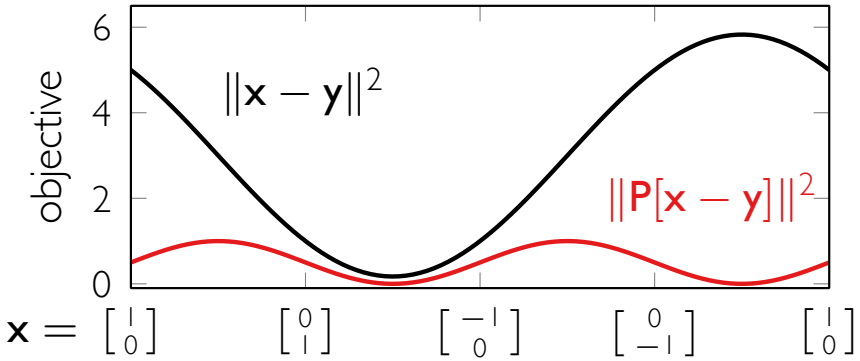

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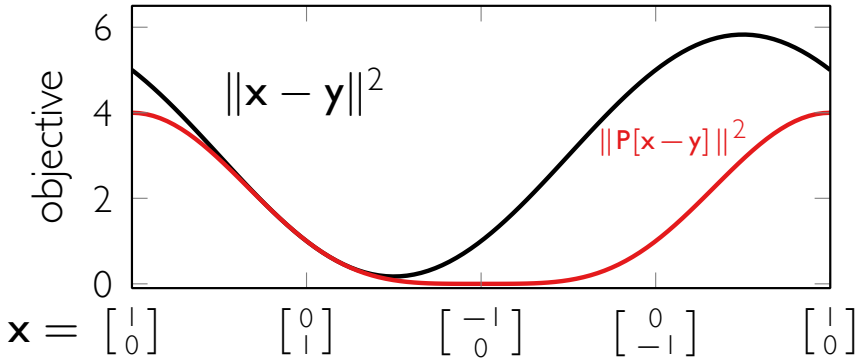

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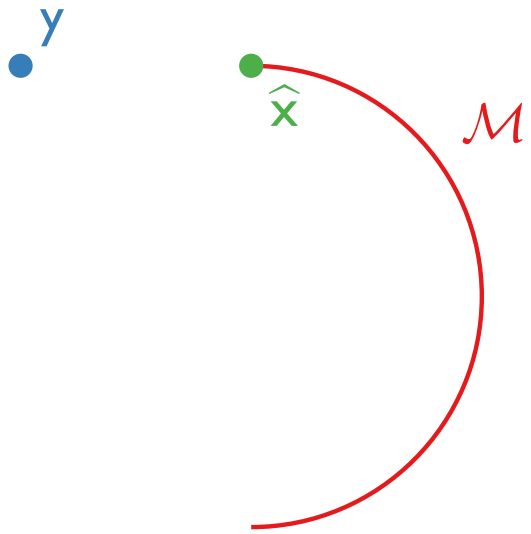

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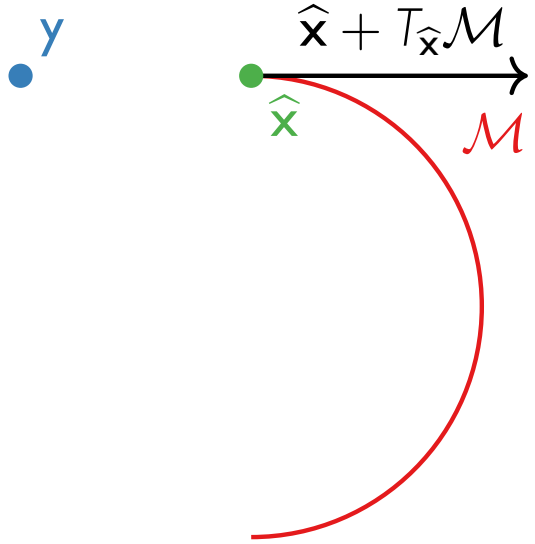

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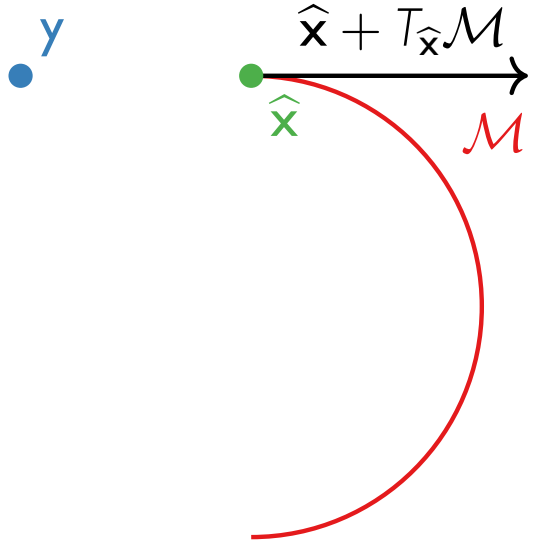

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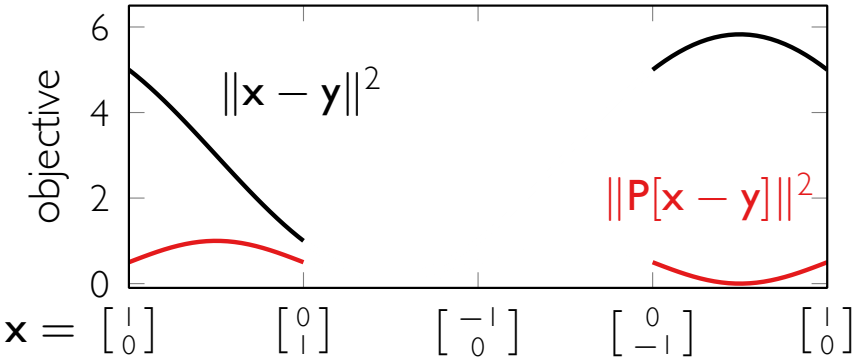

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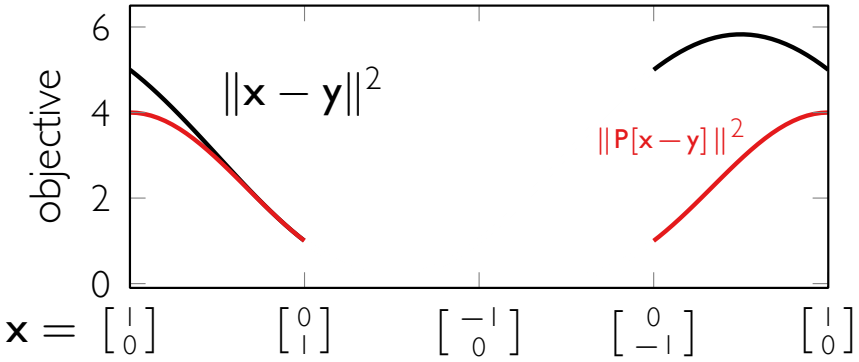

$\mathcal{H}_2$-optimal Model Reduction

Given a stable linear time-invariant system with **impulse response** $h : [0, \infty) \to \mathbb{C}$ mapping **input** $u : [0, \infty) \to \mathbb{C}$ to **output** $y : [0, \infty) \to \mathbb{C}$ via

$$y(t) = \int_0^\tau h(t - \tau) u(t) \, \mathrm{d}t,$$

the **the transfer function** $H \in \mathcal{H}_2$ is the Laplace transform of h; $H = \mathcal{L}h$

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$\mathcal{H}_2$ Hilbert Space

Functions F, G analytic in the right half plane with inner product

$$\langle F, G \rangle_{\mathcal{H}_2} := \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F(i\omega)} G(i\omega) \, \mathrm{d}\omega$$

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Given $H \in \mathcal{H}_2$ and a set $\mathcal{M} \subset \mathcal{H}_2$

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Why use the $\mathcal{H}_2$ norm?

$$\|y - y_r\|_{L_\infty} \leq \|H - H_r\|_{\mathcal{H}_2} \|u\|_{L_2}$$

Constructing a projector exploiting $\mathcal{H}_2$ structure

$\mathcal{H}_2$ is a reproducing kernel Hilbert space (RKHS)

Definition: Primative Vectors

$V[\mu](z) = (z - \mu)^{-1}$, $V[\mu] \in \mathcal{H}_2$ if $\operatorname{Re} \mu < 0$

Function evaluation $\Leftrightarrow$ inner product

$\langle V[\mu], F \rangle_{\mathcal{H}_2} = F(-\overline{\mu}) \quad \langle V[\mu]', F \rangle_{\mathcal{H}_2} = -F'(-\overline{\mu})$

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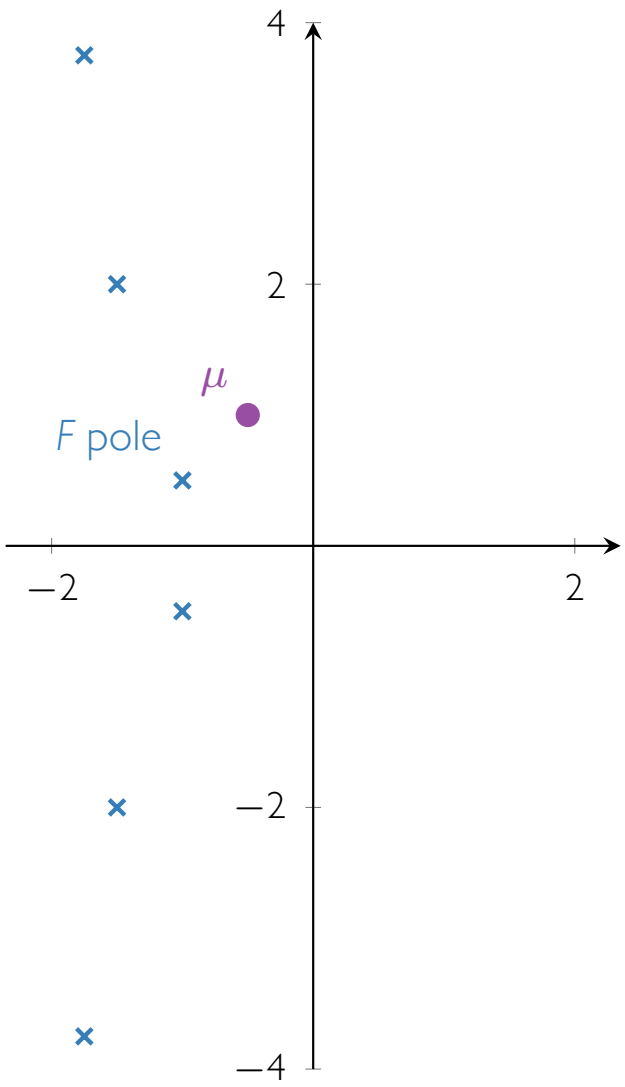

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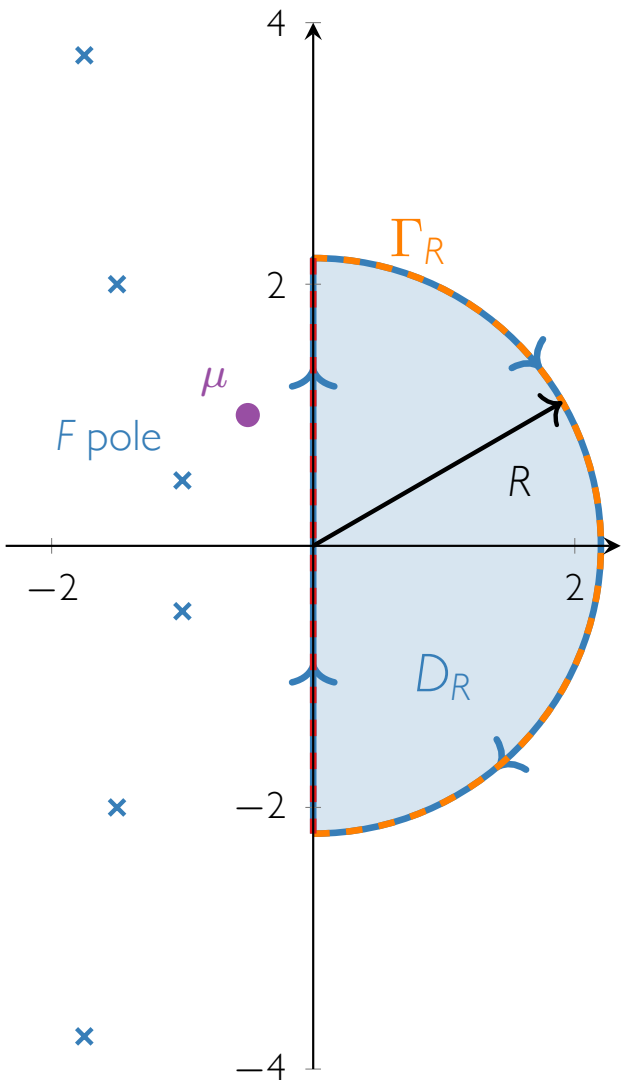

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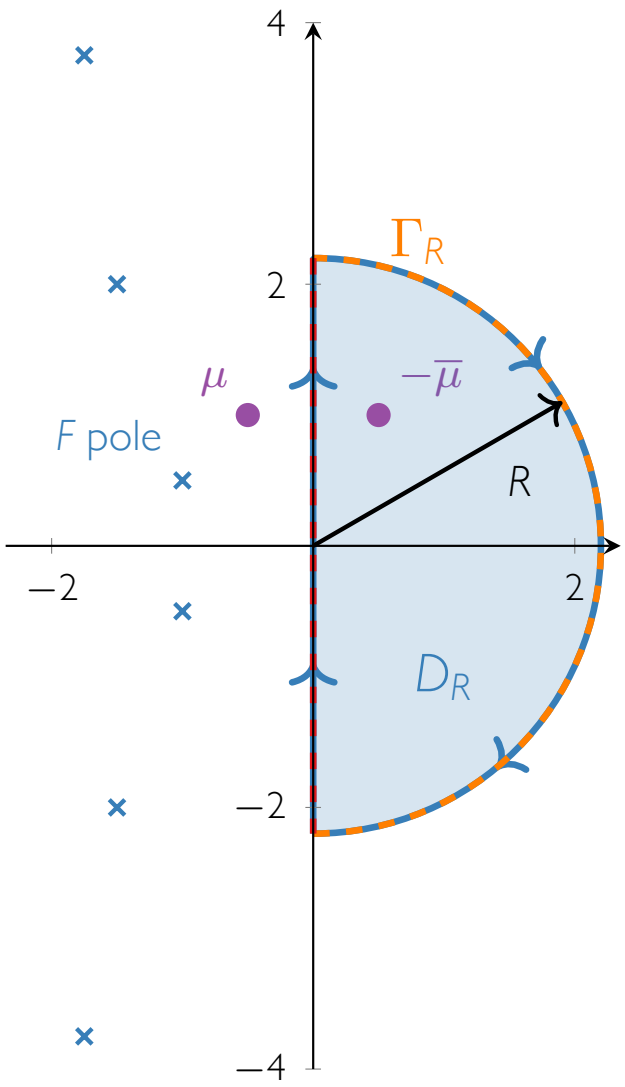

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\end{aligned}
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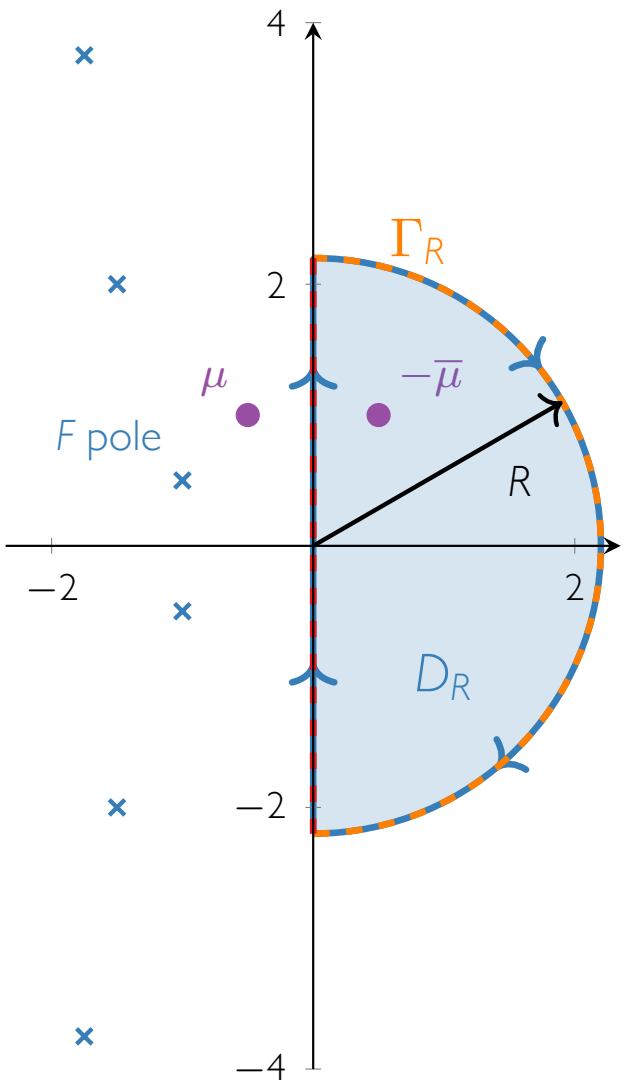

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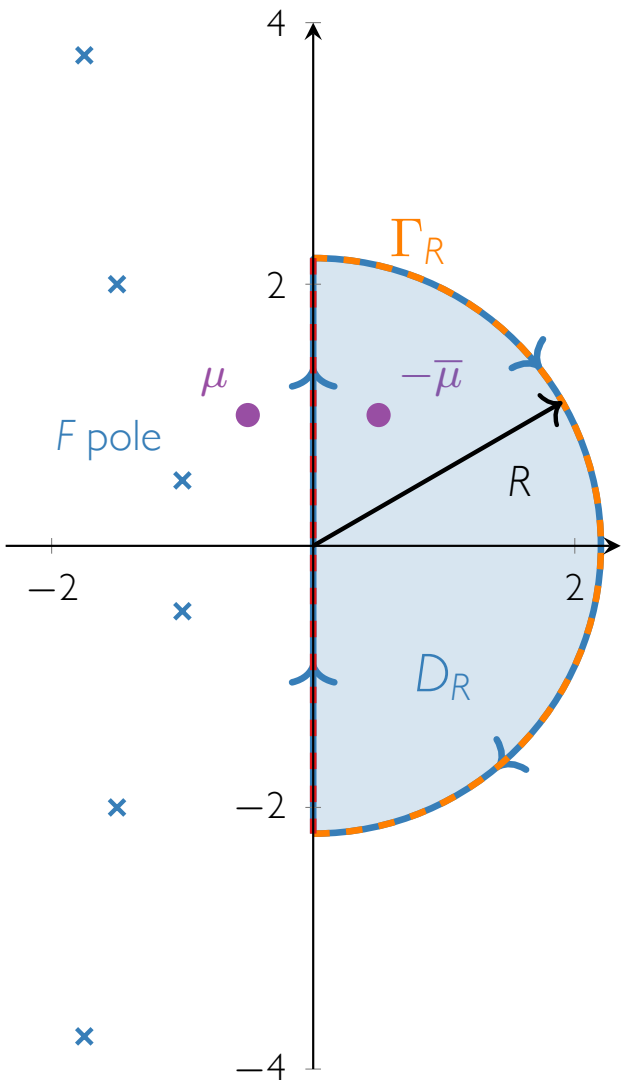
&= \frac{-i}{2\pi}\lim_{R\to\infty}\int_{D_R} (z+\overline{\mu})^{-1}F(z)\,\mathrm{d}z \\
&\quad - \cancelto{0}{\frac{-i}{2\pi}\int_{\Gamma_R} (z+\overline{\mu})^{-1}F(z)\,\mathrm{d}z} \\
&= \operatorname{Res}((z+\overline{\mu})^{-1}F(z); z=-\overline{\mu}) \\
&= F(-\overline{\mu})
\end{aligned}
$$



$\mathcal{H}_2$ is a reproducing kernel Hilbert space (RKHS)

Definition: Primative Vectors

$V[\mu](z) = (z-\mu)^{-1}$, $V[\mu] \in \mathcal{H}_2$ if $\operatorname{Re}\mu < 0$

Function evaluation $\Leftrightarrow$ inner product

$\langle V[\mu], F\rangle_{\mathcal{H}_2} = F(-\overline{\mu}) \quad \langle V[\mu]', F\rangle_{\mathcal{H}_2} = -F'(-\overline{\mu})$

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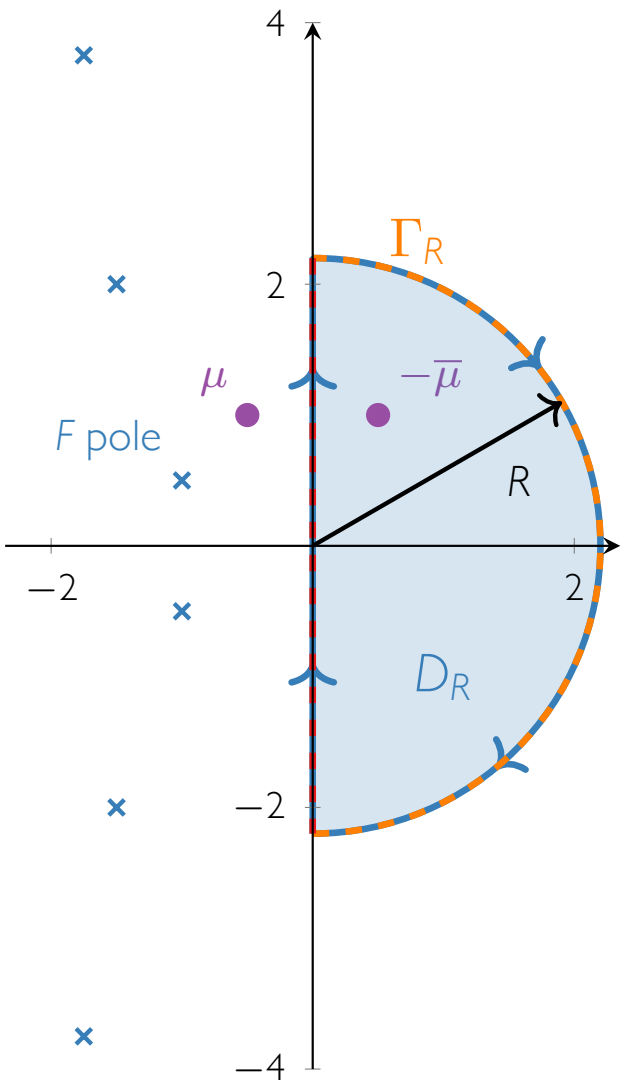

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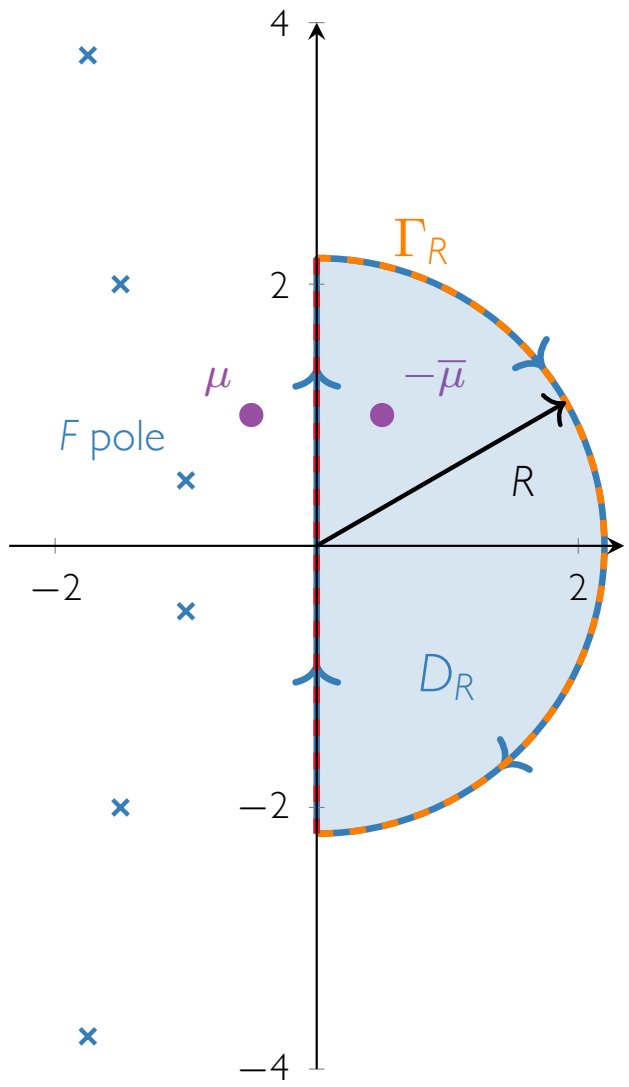

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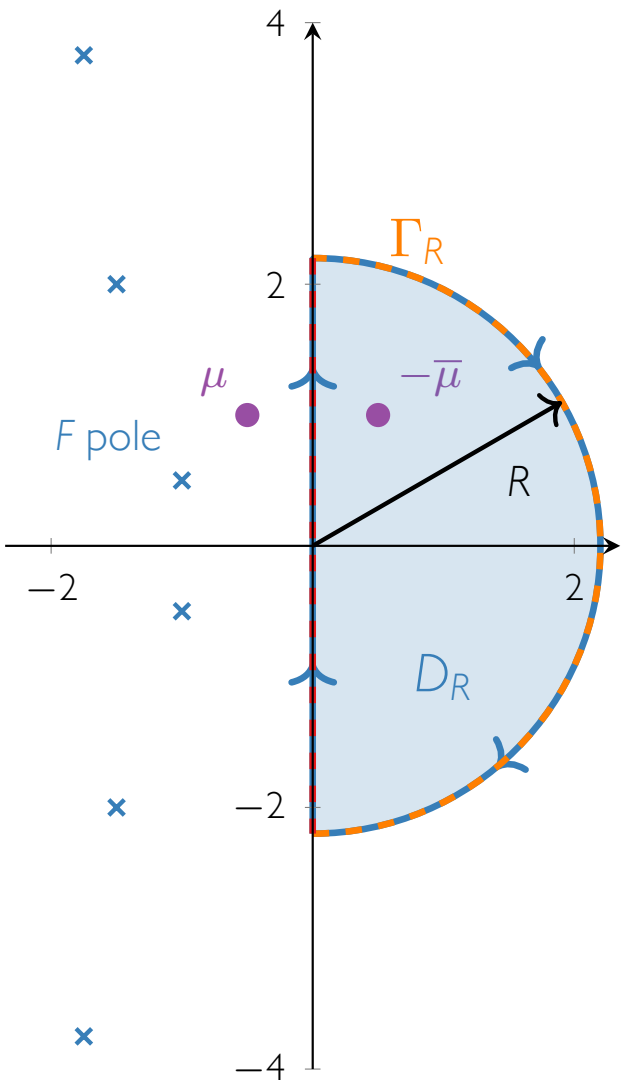

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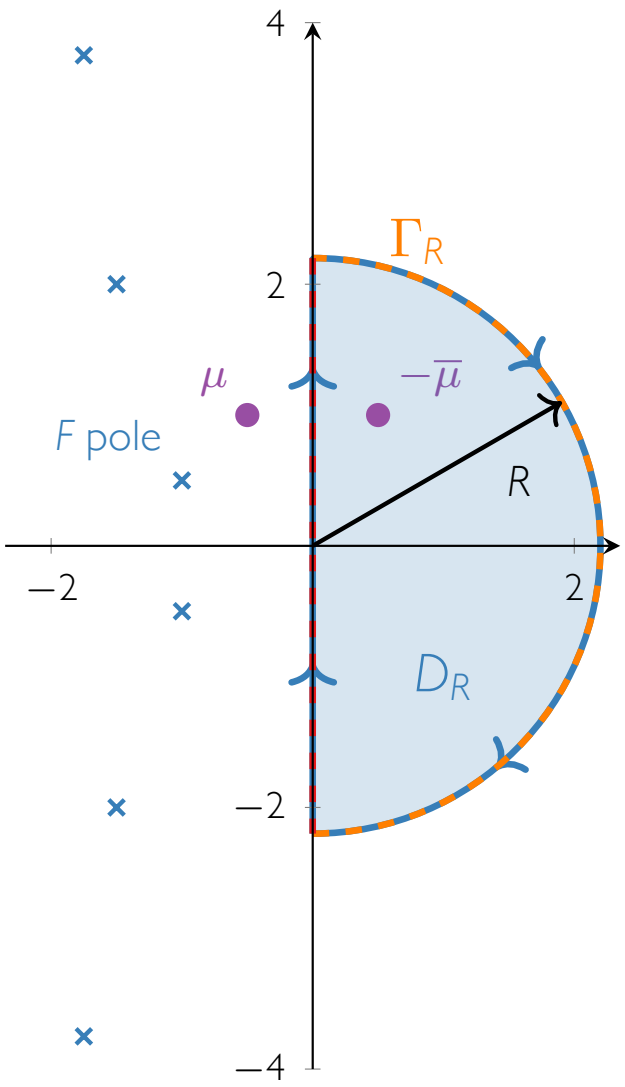

Constructing a projector for $\mathcal{H}_2$ Hilbert space

Idea: Construct a projector from the primatives $V[\mu]$.

Basis Operator $\mathcal{V}(\boldsymbol{\mu})$

$\mathcal{V} : \mathbb{C}^n \to \mathrm{Span}\{V[\mu_j]\}_{j=1}^n$

$$\mathcal{V}(\boldsymbol{\mu})\mathbf{c} := \sum_{j=1}^{n} V[\mu_j]c_j$$

$$\mathcal{V}(\boldsymbol{\mu})^* F = \begin{bmatrix} \langle V[\mu_1], F\rangle_{\mathcal{H}_2} \\ \vdots \\ \langle V[\mu_n], F\rangle_{\mathcal{H}_2} \end{bmatrix} = \begin{bmatrix} F(-\overline{\mu}_1) \\ \vdots \\ F(-\overline{\mu}_n) \end{bmatrix}$$

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Orthogonal Projector $\mathcal{P}(\boldsymbol{\mu})$

$\mathcal{P}(\boldsymbol{\mu}) : \mathcal{H}_2 \to \mathrm{Range}\,\mathcal{V}(\boldsymbol{\mu})$

$$\mathcal{P}(\boldsymbol{\mu}) := \mathcal{V}(\boldsymbol{\mu})[\mathcal{V}(\boldsymbol{\mu})^*\mathcal{V}(\boldsymbol{\mu})]^{-1}\mathcal{V}(\boldsymbol{\mu})^*$$

$$\mathbf{M}(\boldsymbol{\mu}) := \mathcal{V}(\boldsymbol{\mu})^*\mathcal{V}(\boldsymbol{\mu}) \in \mathbb{C}^{n\times n}$$

$$= \begin{bmatrix} \frac{1}{-\overline{\mu}_1 - \mu_1} & \cdots & \frac{1}{-\overline{\mu}_1 - \mu_n} \\ \vdots & & \vdots \\ \frac{1}{-\overline{\mu}_n - \mu_1} & \cdots & \frac{1}{-\overline{\mu}_n - \mu_n} \end{bmatrix}$$

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$$= \langle \mathbf{M}(\boldsymbol{\mu})^{-\frac{1}{2}}\mathcal{V}(\boldsymbol{\mu})^*F, \mathbf{M}(\boldsymbol{\mu})^{-\frac{1}{2}}\mathcal{V}(\boldsymbol{\mu})^*F\rangle_2 = \left\| \mathbf{M}(\boldsymbol{\mu})^{-\frac{1}{2}} \begin{bmatrix} F(-\overline{\mu}_1) \\ \vdots \\ F(-\overline{\mu}_n) \end{bmatrix} \right\|_2^2$$

Projected Problem for $\mathcal{H}_2$-optimal Model Reduction

$$\min_{H_r \in \mathcal{M}} \|\mathcal{P}(\boldsymbol{\mu})[H - H_r]\|_{\mathcal{H}_2}^2, \qquad \|\mathcal{P}(\boldsymbol{\mu})[H - H_r]\|_{\mathcal{H}_2}^2 = \left\| \mathbf{M}(\boldsymbol{\mu})^{-\frac{1}{2}} \begin{bmatrix} H(-\overline{\mu}_1) - H_r(-\overline{\mu}_1) \\ \vdots \\ H(-\overline{\mu}_n) - H_r(-\overline{\mu}_n) \end{bmatrix} \right\|_2^2$$

How do we choose projector $\mathcal{P}(\boldsymbol{\mu})$?

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Recall Projection lemma requires Range $\mathcal{P}(\boldsymbol{\mu}) \supseteq T_{H_r}\mathcal{M}$ where $T_{H_r}\mathcal{M}$ is tangent cone

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Question What is $\mathcal{M}$?

Space of Reduced Order Models $\mathcal{M}$

State-Space Reduced Order Models

$$\mathcal{S}_r(\mathbb{C}) := \{H \in \mathcal{H}_2 : H(z) = \mathbf{c}^*(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},\ \mathbf{b}, \mathbf{c} \in \mathbb{C}^r,\ \mathbf{A} \in \mathbb{C}^{r\times r}\} \qquad \textbf{complex}$$

$$\mathcal{S}_r(\mathbb{R}) := \{H \in \mathcal{H}_2 : H(z) = \mathbf{c}^\top(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b},\ \mathbf{b}, \mathbf{c} \in \mathbb{R}^r,\ \mathbf{A} \in \mathbb{R}^{r\times r}\} \qquad \textbf{real}$$

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If H_r has only simple poles, then $\mathbf{A}$ is diagonalizable $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$

$$\begin{aligned} H_r(z) &= \mathbf{c}^*(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = \mathbf{c}^*(z\mathbf{I} - \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1})^{-1}\mathbf{b} = \mathbf{c}^*\mathbf{V}(z\mathbf{I} - \mathbf{\Lambda})^{-1}\mathbf{V}^{-1}\mathbf{b} \\ &= \sum_{k=1}^{r}[\mathbf{c}^*\mathbf{V}]_k[\mathbf{V}^{-1}\mathbf{b}]_k(z - \lambda_k)^{-1} = \sum_{k=1}^{r}\rho_k V[\lambda_k](z) \end{aligned}$$

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$$\mathcal{S}_r(\mathbb{C}) = \left\{\sum_{k=1}^{r}\rho_k V[\lambda_k] : \rho_k, \lambda_k \in \mathbb{C},\ \operatorname{Re}\lambda_k < 0\right\}$$
$$\cup \{H \in \mathcal{S}_r(\mathbb{C}) : H \text{ has pole of multiplicity} \geq 2\}$$

Note H_r with higher order poles are nowhere dense in $\mathcal{H}_2$

Tangent Cone of $\mathcal{S}_r(\mathbb{C})$

If H_r is has only simple poles, then $H_r = \sum_{k=1}^{r} \rho_k V[\lambda_k]$ and

$$T_{H_r}\mathcal{S}_r(\mathbb{C}) = \mathrm{Span}\{V[\lambda_k], V[\lambda_k]'\}_{k=1}^{r}$$

Tangent cone $\quad T_x\mathcal{M} := \{t : \exists \{\epsilon_\ell\}_{\ell=1}^{\infty} \geq 0,\ \{x_\ell\}_{\ell=1}^{\infty} \subset \mathcal{M},\ x_\ell \to x,\ \epsilon_\ell(x_\ell - x) \to t\}$

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Pole-residue form $\quad H_r = \sum_{k=1}^{r} \rho_k V[\lambda_k],$ with $\rho_k, \lambda_k \in \mathbb{C}$, $\mathrm{Re}\,\lambda_k < 0$

Span of $V[\lambda_k]$

Pick $\theta \in [0, 2\pi)$, $\delta_\ell \to 0$, $\delta_\ell \geq 0$ set

$$H_r^\ell = H_r + e^{i\theta}\delta_\ell V[\lambda_k] \in \mathcal{S}_r(\mathbb{C})$$

$$\epsilon_\ell = \alpha\delta_\ell^{-1} \quad \alpha \geq 0$$

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Tangent vectors:

$$\lim_{\ell\to\infty} \epsilon_\ell(H_r^\ell - H_r) = \alpha \lim_{\ell\to\infty} \frac{e^{i\theta}\delta_\ell}{\delta_\ell} V[\lambda_k]$$

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Span of $V[\lambda_k]'$

Pick $\theta \in [0, 2\pi)$, $\delta_\ell \to 0$, $\delta_\ell \geq 0$ set

$$H_r^\ell = H_r + \rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k] \in \mathcal{S}_r(\mathbb{C})$$

$$\epsilon_\ell = \alpha \|\rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k]\|_{\mathcal{H}_2}^{-1} \quad \alpha \geq 0$$

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$$\epsilon_\ell = \alpha\delta_\ell^{-1} \quad \alpha \geq 0$$

Tangent vectors:

$$\lim_{\ell\to\infty} \epsilon_\ell(H_r^\ell - H_r) = \alpha \lim_{\ell\to\infty} \frac{e^{i\theta}\delta_\ell}{\delta_\ell} V[\lambda_k]$$
$$= \alpha e^{i\theta} V[\lambda_k]$$

Span of $V[\lambda_k]'$

Pick $\theta \in [0, 2\pi)$, $\delta_\ell \to 0$, $\delta_\ell \geq 0$ set

$$H_r^\ell = H_r + \rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k] \in \mathcal{S}_r(\mathbb{C})$$

$$\epsilon_\ell = \alpha \|\rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k]\|_{\mathcal{H}_2}^{-1} \quad \alpha \geq 0$$

Tangent vectors:

$$\lim_{\ell\to\infty} \epsilon_\ell(H_r^\ell - H_r) = \lim_{\ell\to\infty} \frac{\alpha(\rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k])}{\|\rho_k V[\lambda_k + e^{i\theta}\delta_\ell] - \rho_k V[\lambda_k]\|_{\mathcal{H}_2}}$$
$$= \alpha\rho_k V[\lambda_k]' e^{i\theta}$$

Tangent Cone of $\mathcal{S}_r(\mathbb{C})$

If H_r is has only simple poles, then $H_r = \sum_{k=1}^{r} \rho_k V[\lambda_k]$ and

$$T_{H_r}\mathcal{S}_r(\mathbb{C}) = \text{Span}\{V[\lambda_k], V[\lambda_k]'\}_{k=1}^{r}$$

Tangent cone $\quad T_x\mathcal{M} := \{t : \exists \{\epsilon_\ell\}_{\ell=1}^{\infty} \geq 0,\ \{x_\ell\}_{\ell=1}^{\infty} \subset \mathcal{M},\ x_\ell \to x,\ \epsilon_\ell(x_\ell - x) \to t\}$

Pole-residue form $\quad H_r = \sum_{k=1}^{r} \rho_k V[\lambda_k],$ with $\rho_k, \lambda_k \in \mathbb{C}$, $\text{Re}\,\lambda_k < 0$

Span of $V[\lambda_k]$

Pick $\theta \in [0, 2\pi)$, $\delta_\ell \to 0$, $\delta_\ell \geq 0$ set

$$H_r^\ell = H_r + e^{i\theta}\delta_\ell V[\lambda_k] \in \mathcal{S}_r(\mathbb{C})$$

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$$= \alpha\rho_k V[\lambda_k]' e^{i\theta}$$

As $\mathcal{S}_r(\mathbb{R}) \subset \mathcal{S}_r(\mathbb{C})$, if $H_r \in \mathcal{S}_r(\mathbb{R})$ then $T_{H_r}\mathcal{S}_r(\mathbb{R}) \subset T_{H_r}\mathcal{S}_r(\mathbb{C})$

Building a practical projected algorithm

Transfer Function IRKA

[Beattie & Gugercin 09]

Idea: Choose Range $\mathcal{P}$ to be tangent cone $\mathcal{T}_{H_r}\mathcal{S}_r(\mathbb{C})$ at previous iteration

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for $\ell = 0, 1, 2, \ldots$ **do**

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Recall $T_{H_r}\mathcal{S}_r(\mathbb{C}) = \{V[\lambda_k], V[\lambda_k]'\}_{k=1}^r$

$$\|\widehat{\mathcal{P}}(\boldsymbol{\lambda})[H - H_r]\|_{\mathcal{H}_2}^2 =$$

$$\left\| \widehat{\mathbf{M}}(\boldsymbol{\lambda})^{-\frac{1}{2}} \begin{bmatrix} H(-\overline{\lambda}_1) - H_r(-\overline{\lambda}_1) \\ \vdots \\ H(-\overline{\lambda}_r) - H_r(-\overline{\lambda}_r) \\ -H'(-\overline{\lambda}_1) + H_r'(-\overline{\lambda}_1) \\ \vdots \\ -H'(-\overline{\lambda}_1) + H_r'(-\overline{\lambda}_1) \end{bmatrix} \right\|_2^2$$

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- **if** $\|H_r^\ell - H_r^{\ell-1}\|_{\mathcal{H}_2} \leq \tau$ **then break**;

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have $2r$ complex data
H_r has $2r$ complex parameters
$\Rightarrow$ construct **interpolant** where

$$H(-\overline{\lambda}) = H_r(-\overline{\lambda}) \quad H'(-\overline{\lambda}) = H_r'(-\overline{\lambda})$$

As the error is zero, $\widehat{\mathbf{M}}(\boldsymbol{\lambda})$ **ignored**

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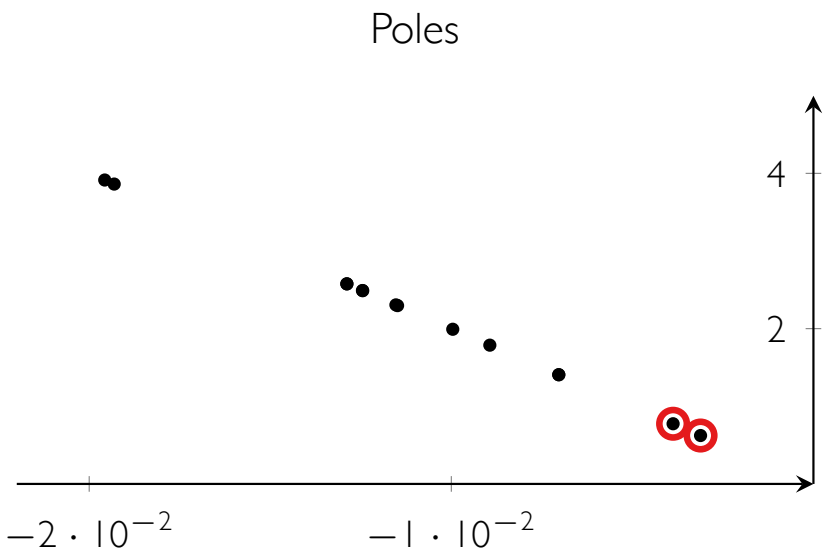

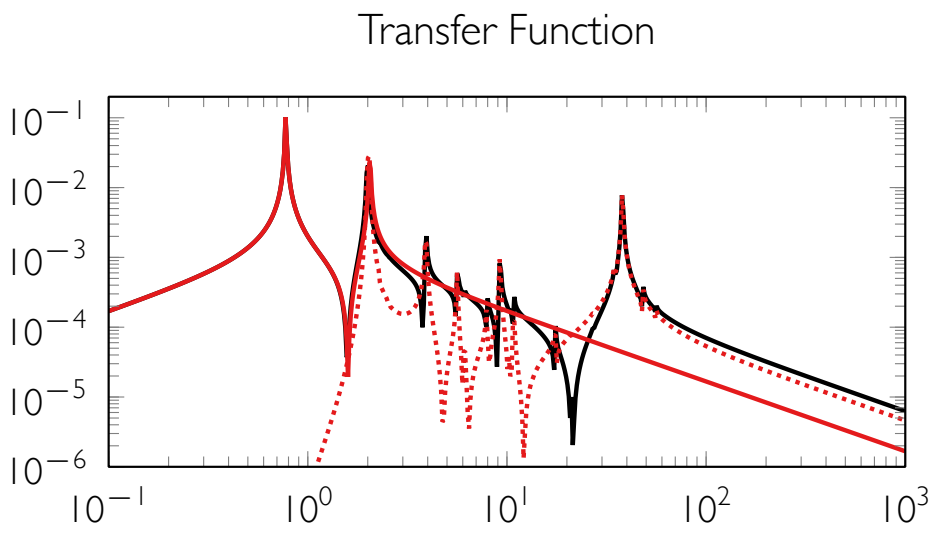

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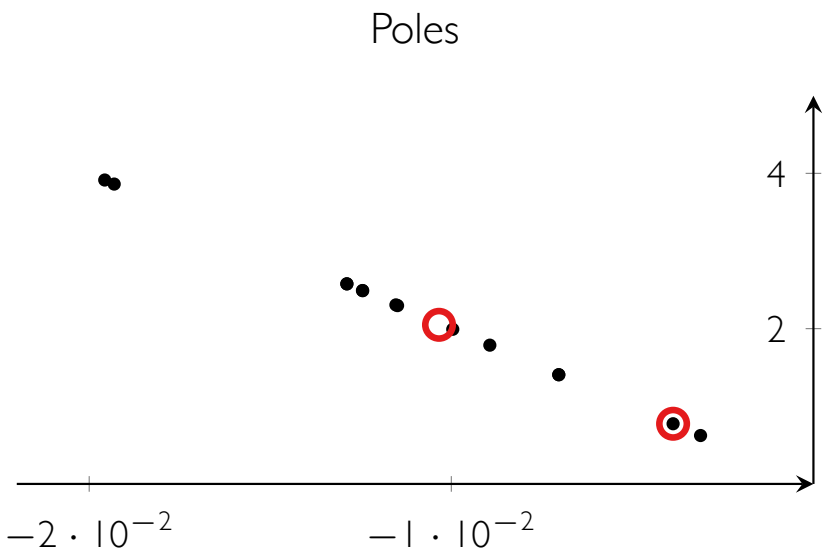

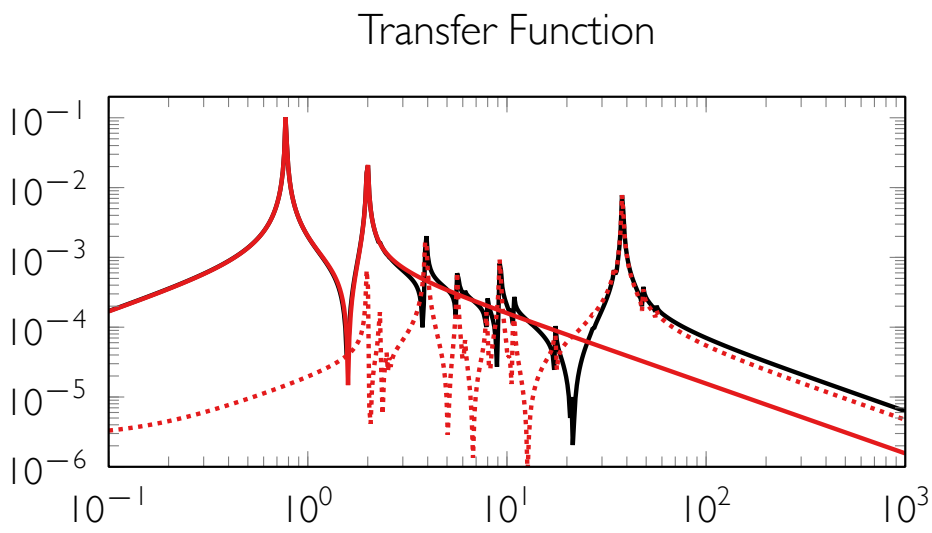

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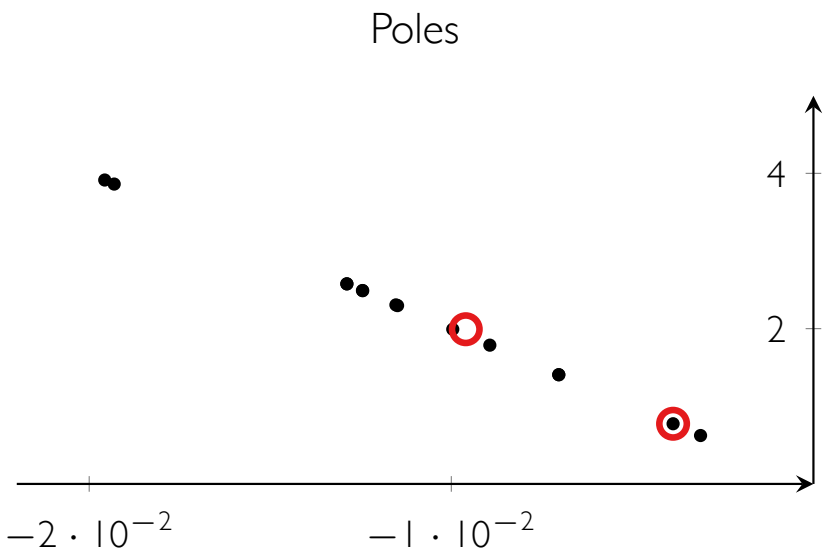

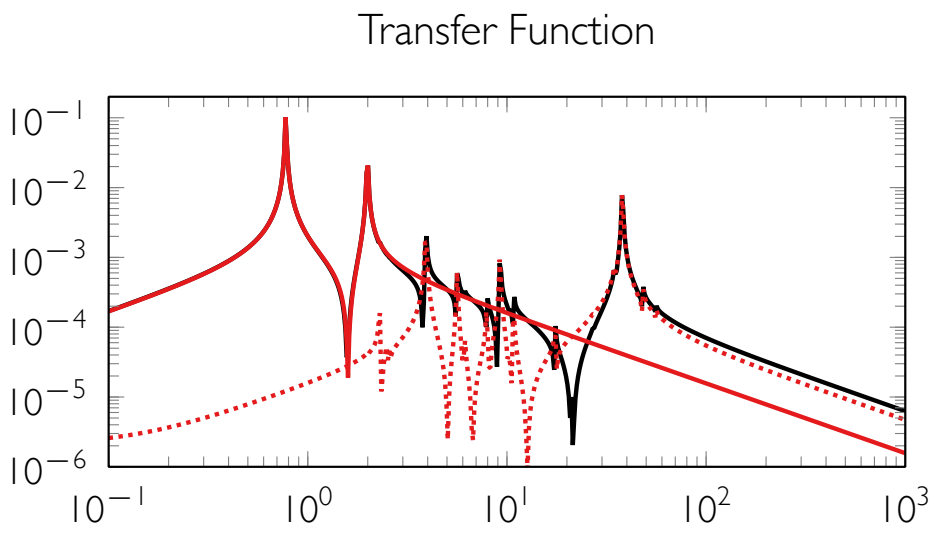

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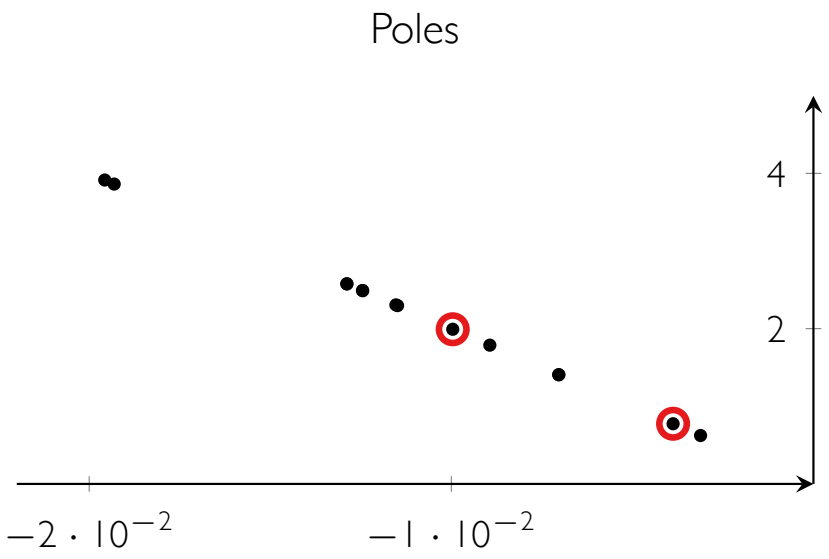

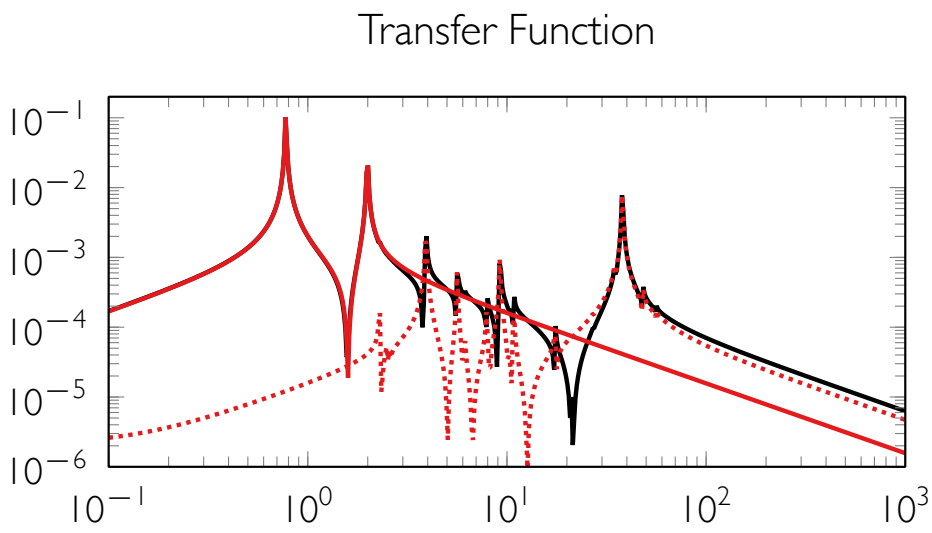

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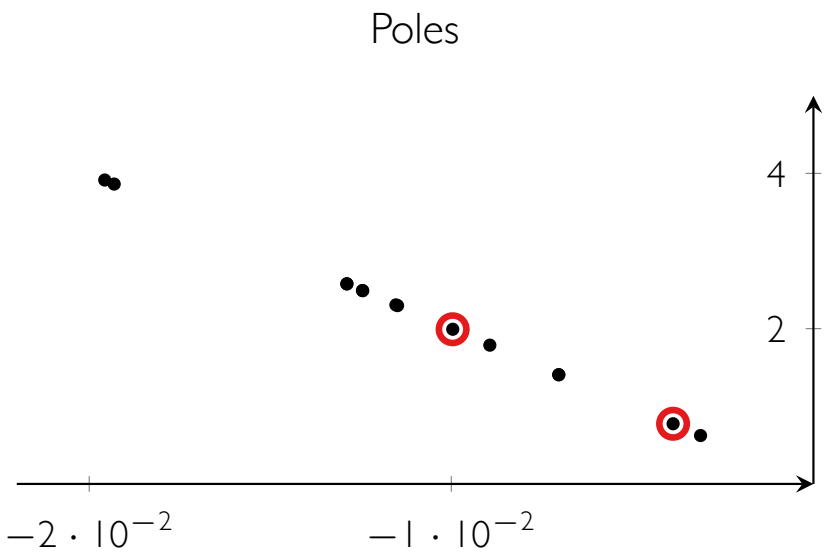
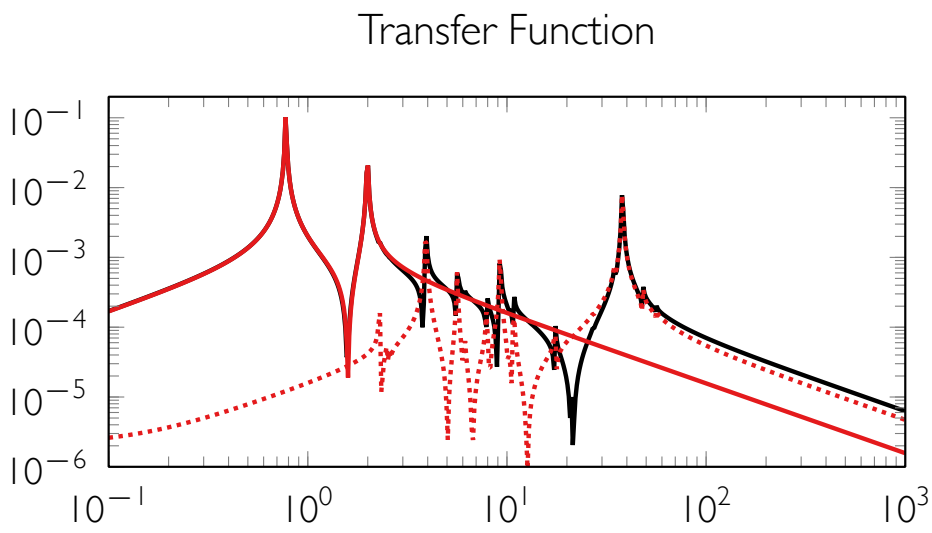
- **Discards** previous samples $H(-\overline{\lambda}_k^\ell), H(-\overline{\lambda}_k^\ell)$





A more efficient projected algorithm

Projected $\mathcal{H}_2$

[H. & Magruder]

Idea: Increase Range $\mathcal{P}$ at each step to better contain tangent cone $\mathcal{T}_{H_r}\mathcal{S}_r(\mathbb{C})$

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PH2

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Projected $\mathcal{H}_2$

[H. & Magruder]

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Recall $T_{H_r}\mathcal{S}_r(\mathbb{C}) = \{V[\lambda_k], V[\lambda_k]'\}_{k=1}^r$

Although $\mathcal{P}(\boldsymbol{\mu}) \not\supseteq T_{H_r}\mathcal{S}_r(\mathbb{C})$,
can get **arbitrarly close**

$$\mathrm{Span}\{V[\lambda], V[\lambda]'\} \subsetneq \mathrm{Span}\{V[\lambda - \delta], V[\lambda + \delta]\}$$

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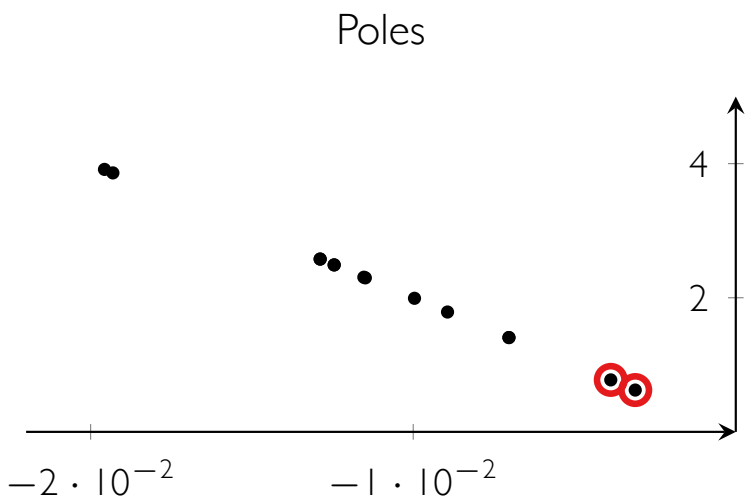

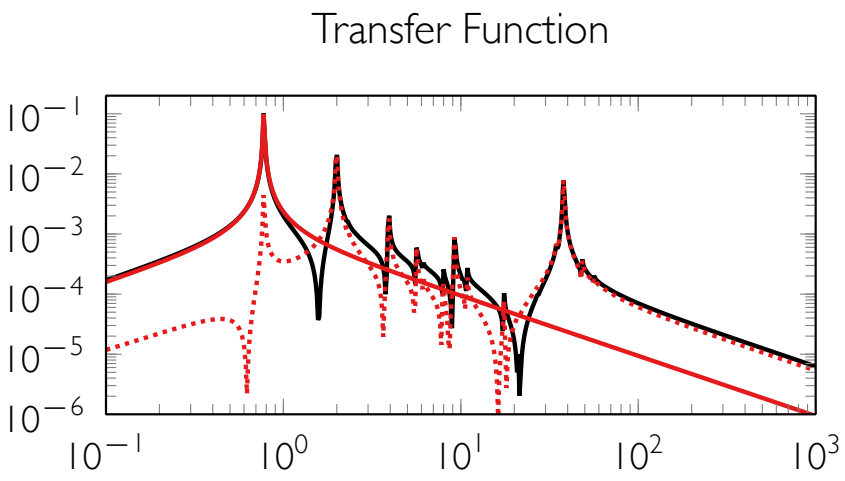

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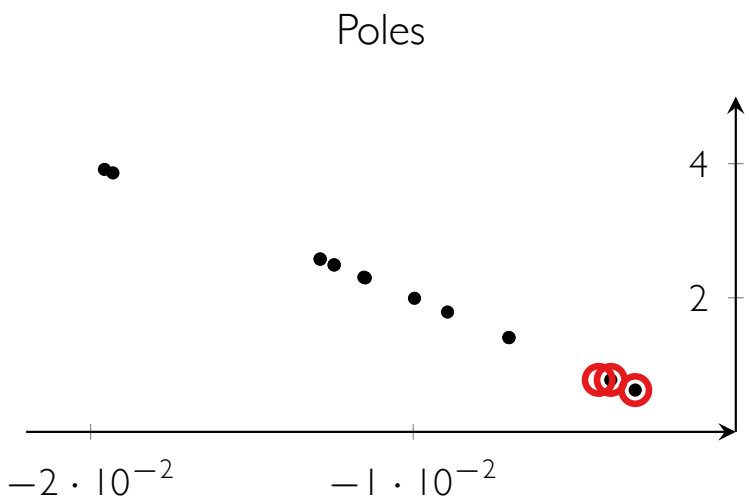

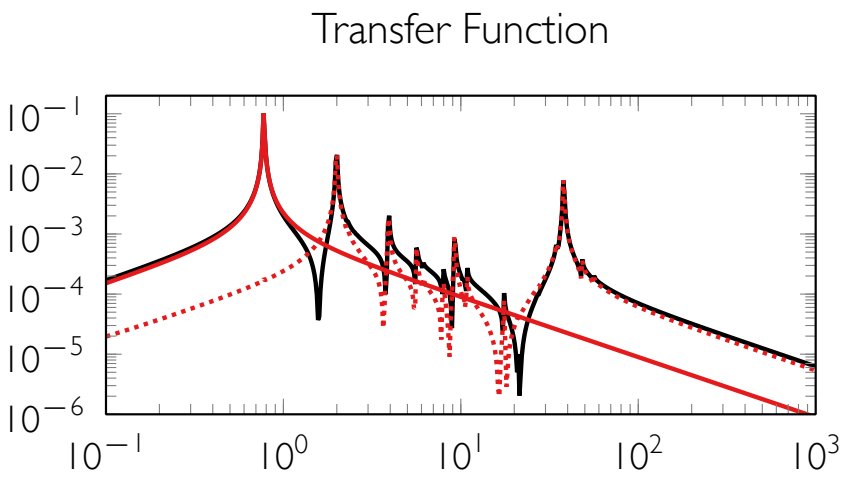

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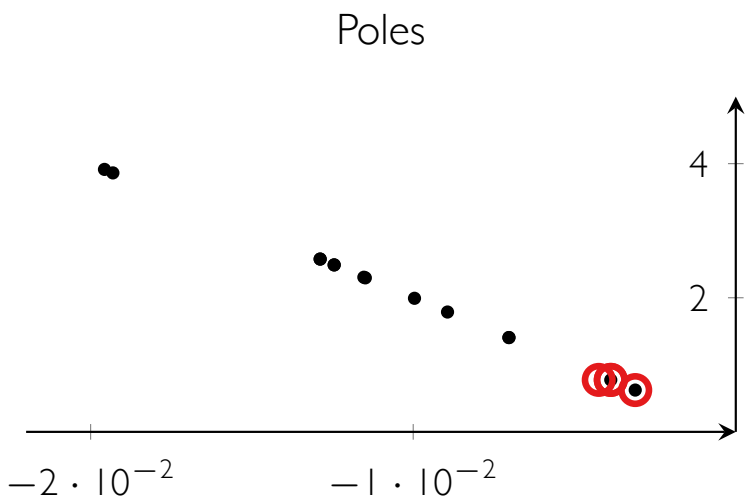

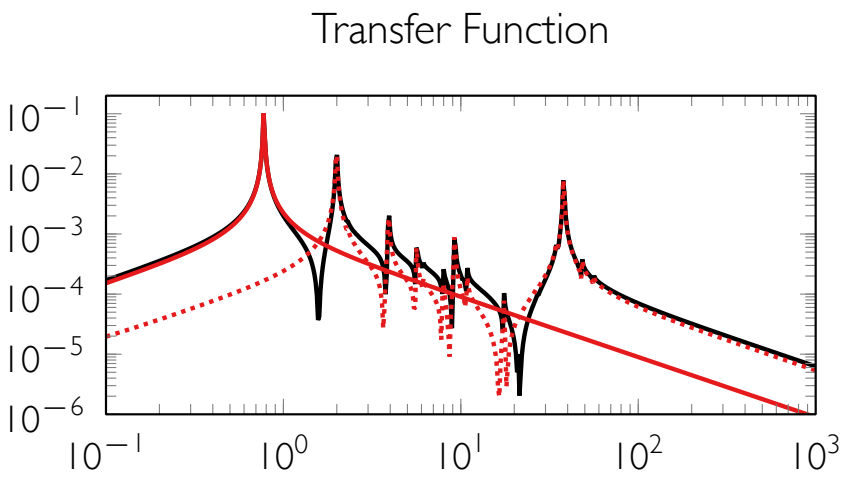

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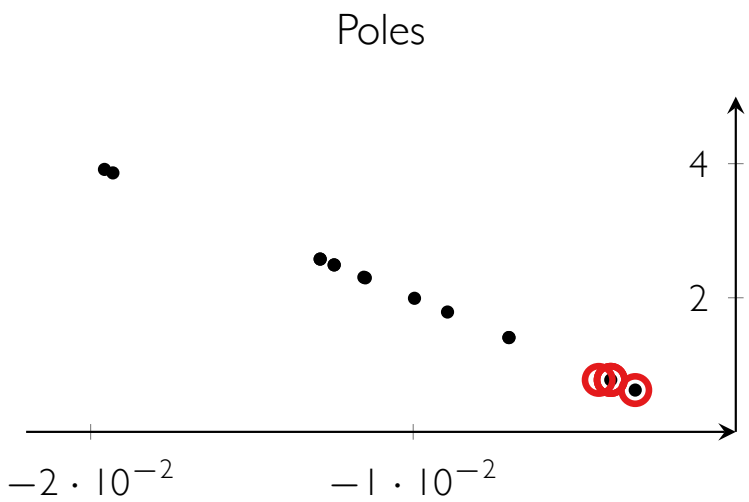

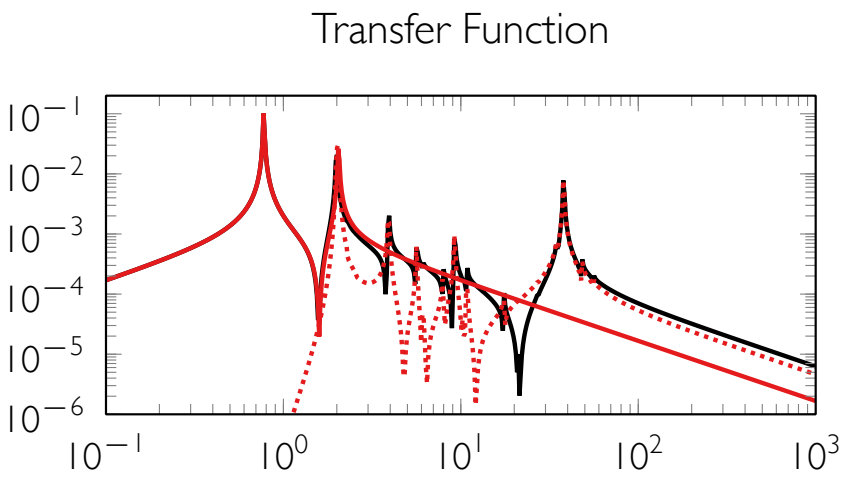

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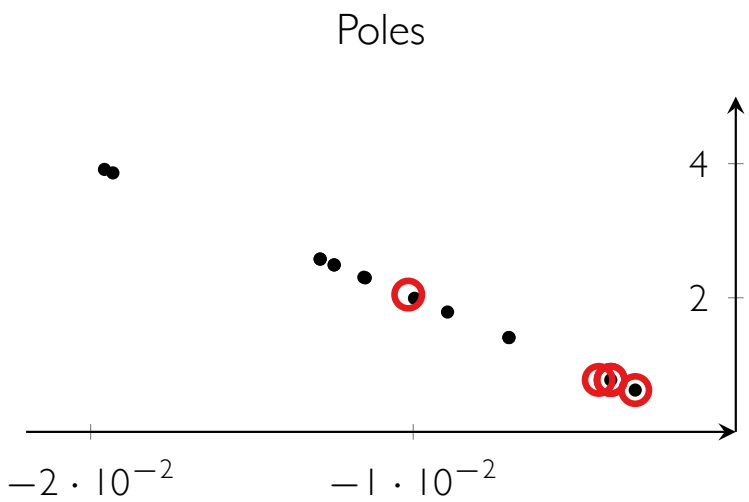

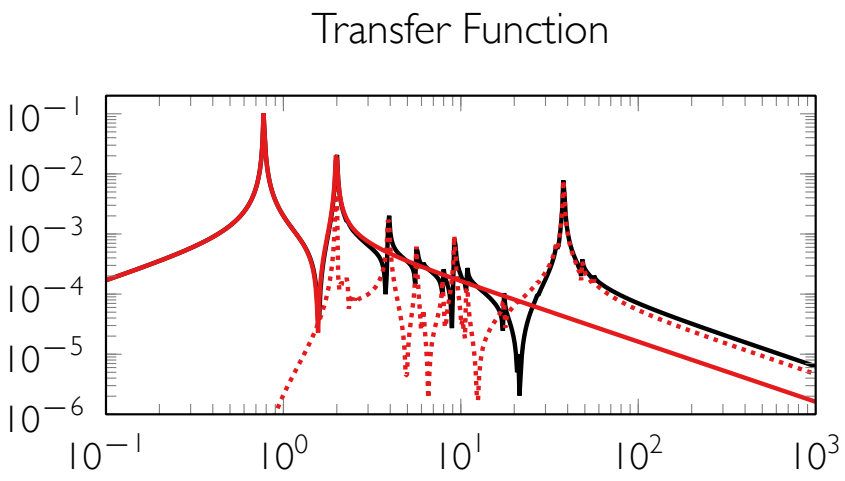

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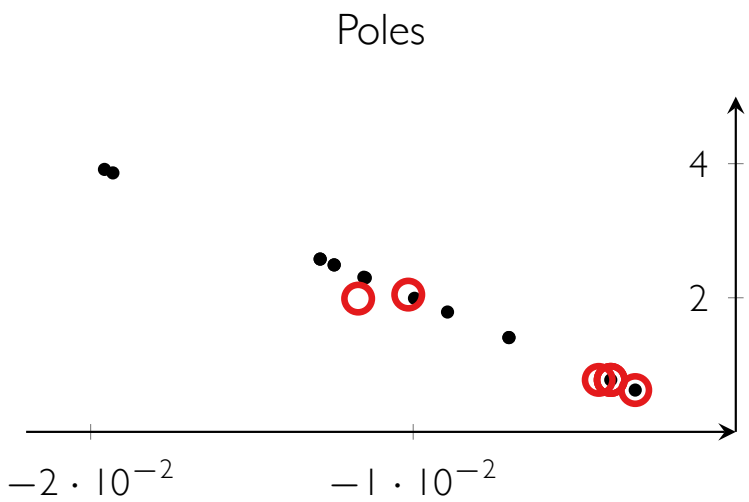

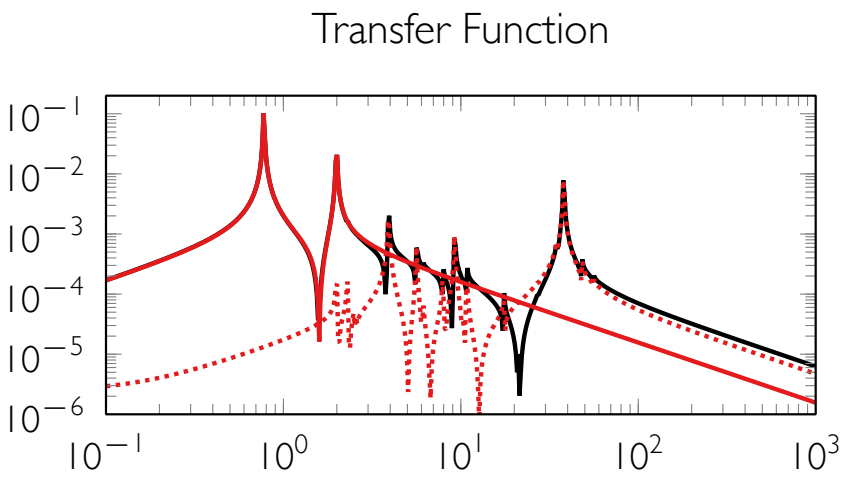

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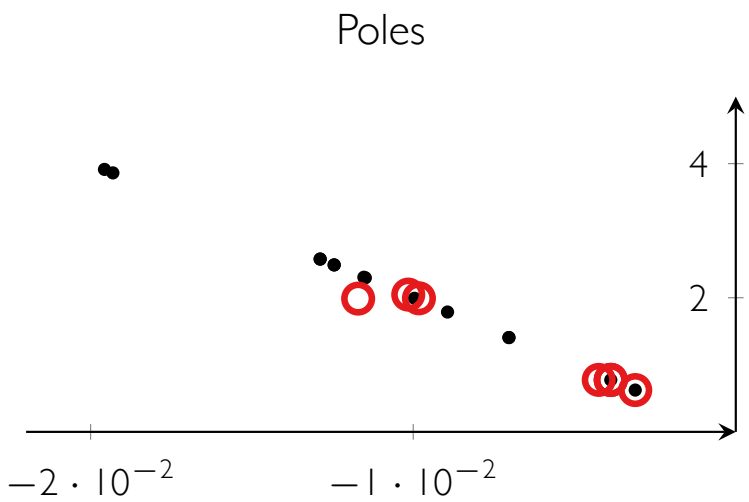

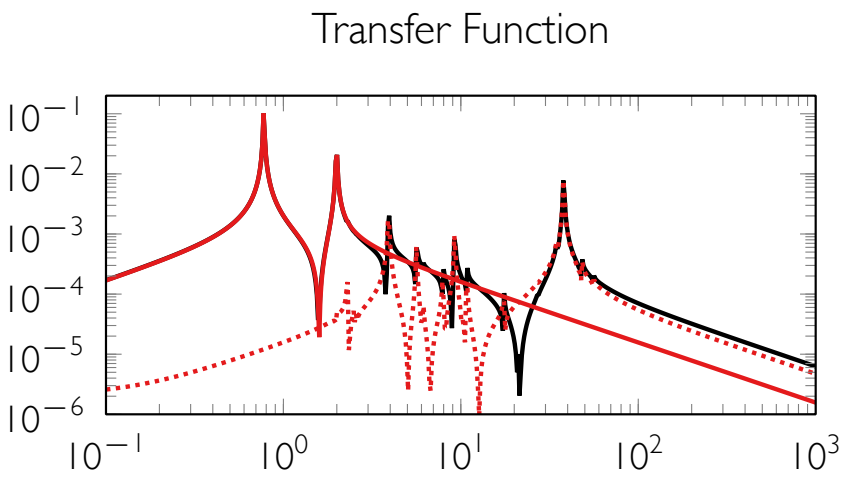

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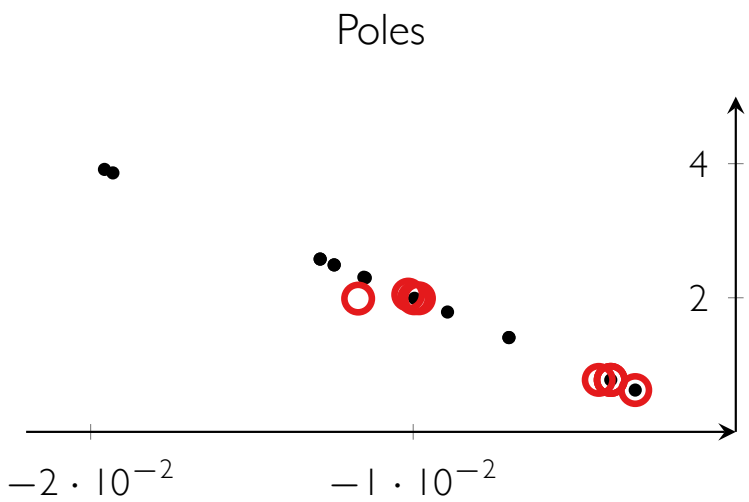
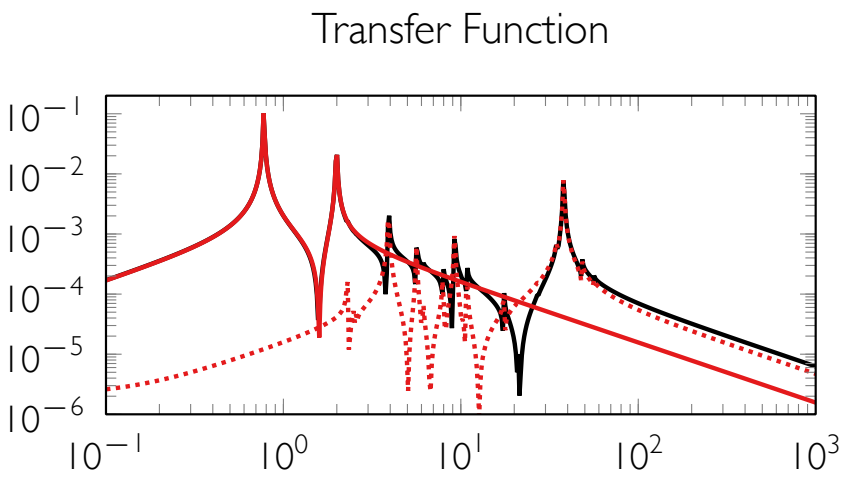




Projected $\mathcal{H}_2$

[H. & Magruder]

Idea: Increase Range $\mathcal{P}$ at each step to better contain tangent cone $\mathcal{T}_{H_r}\mathcal{S}_r(\mathbb{C})$

PH2

Input : $H \in \mathcal{H}_2$, initial samples $\boldsymbol{\mu}^0$
Output : $H_r \in \mathcal{S}_r(\mathbb{C})$
for $\ell = 0, 1, 2, \ldots$ **do**

$\mathcal{P}(\boldsymbol{\mu}^\ell) : \mathcal{H}_2 \to \mathrm{Span}\{V[\mu_k^\ell]\}_k$;

$H_r^\ell \leftarrow \min_{H_r \in \mathcal{S}_r(\mathbb{C})} \|\mathcal{P}(\boldsymbol{\mu}^\ell)[H - H_r]\|_{\mathcal{H}_2}$;

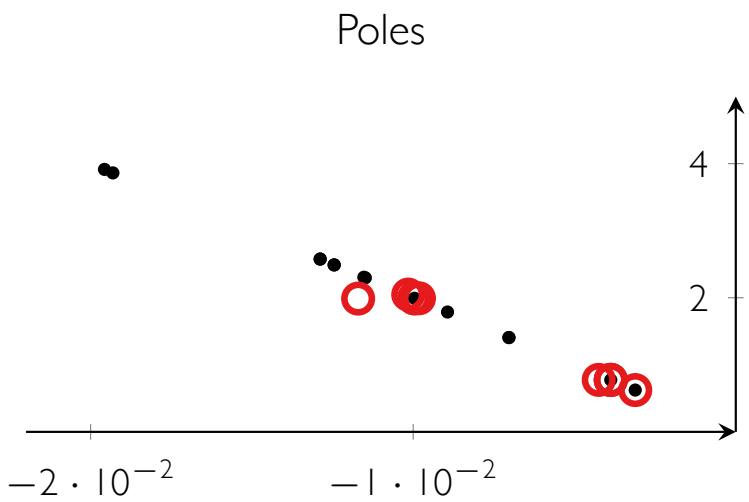
$\boldsymbol{\lambda} \leftarrow$ poles of H_r^ℓ;

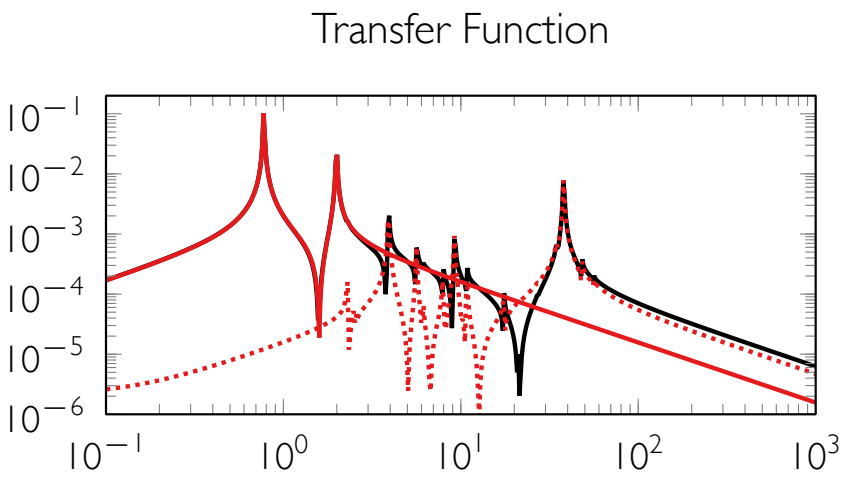
$\mu_\star \leftarrow \operatorname{argmax}_{\lambda \in \boldsymbol{\lambda}} \measuredangle(\mathrm{Range}\,\mathcal{P}(\boldsymbol{\mu}), T_{V[\lambda]}\mathcal{S}_1(\mathbb{C}))$;

$\boldsymbol{\mu}^{\ell+1} \leftarrow [\boldsymbol{\mu}^\ell, \mu_\star]$;

if $\|H_r^\ell - H_r^{\ell-1}\|_{\mathcal{H}_2} \leq \tau$ **then break**;

- Overdetermined **approximation** problem
- If underdetermined, uses smaller rank r
- Select $\mu_\star$ based on **subspace angle** $\measuredangle$
- Converges under mild conditions





Theoretical Results

Thm: Convergence of Projected $\mathcal{H}_2$ Iteration [H. & Magruder]

Let $H \in \mathcal{H}_2$, $n_0, r \in \mathbb{Z}_+$ with $n_0 \geq 2r$. Given $\boldsymbol{\mu}^0 \in \mathbb{C}_-^{n_0}$ with distinct entries, generate

$$H_r^\ell = \operatorname*{argmin}_{H_r \in \mathcal{S}_r(\mathbb{C})} \|\mathcal{P}(\boldsymbol{\mu}^\ell)[H - H_r]\|_{\mathcal{H}_2}$$

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If $\{\mu_\star^\ell\}_{\ell=0}^\infty$ has distinct bounded entries and $H_r^\ell \to H_r \in \mathcal{H}_2$ where H_r has r distinct poles then, H_r is a local optimizer to $\|H - H_r\|_{\mathcal{H}_2}$.

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- As $\{\mu_\star^\ell\}_{\ell=0}^\infty$ bounded, then has at least one limit point
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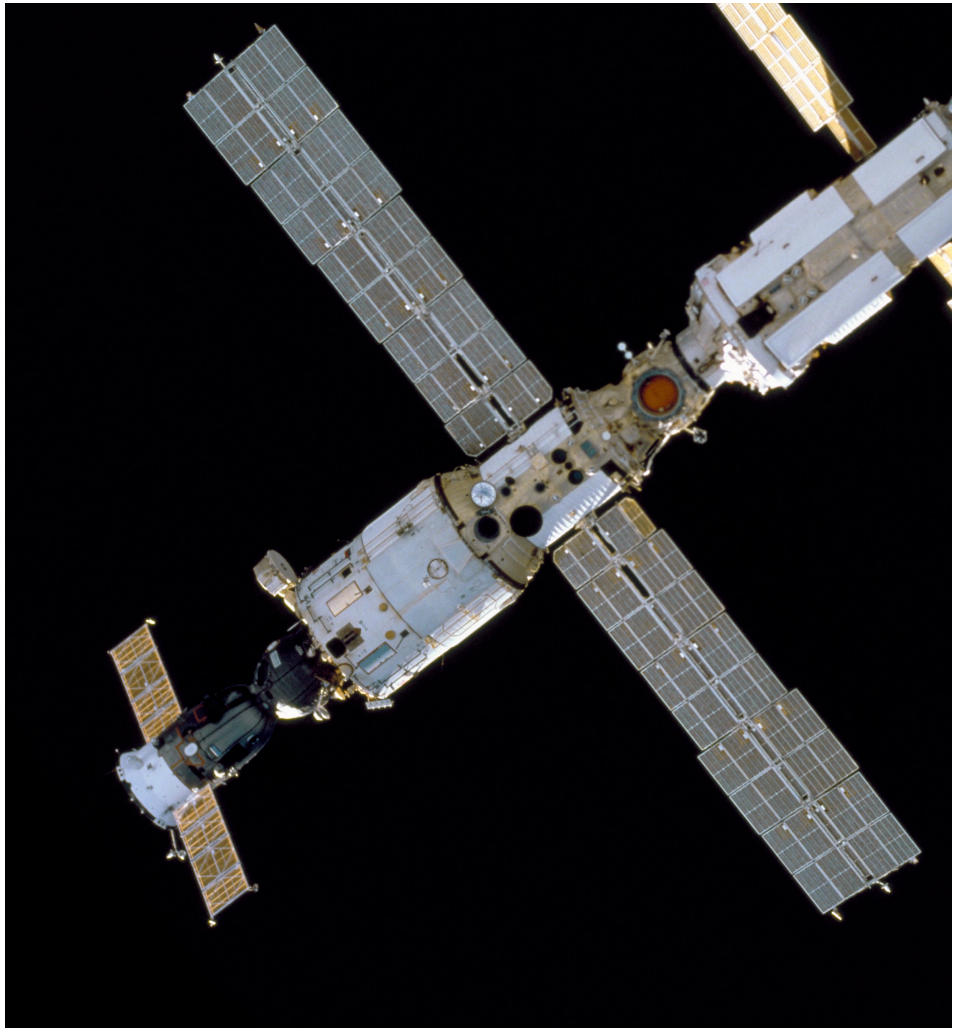
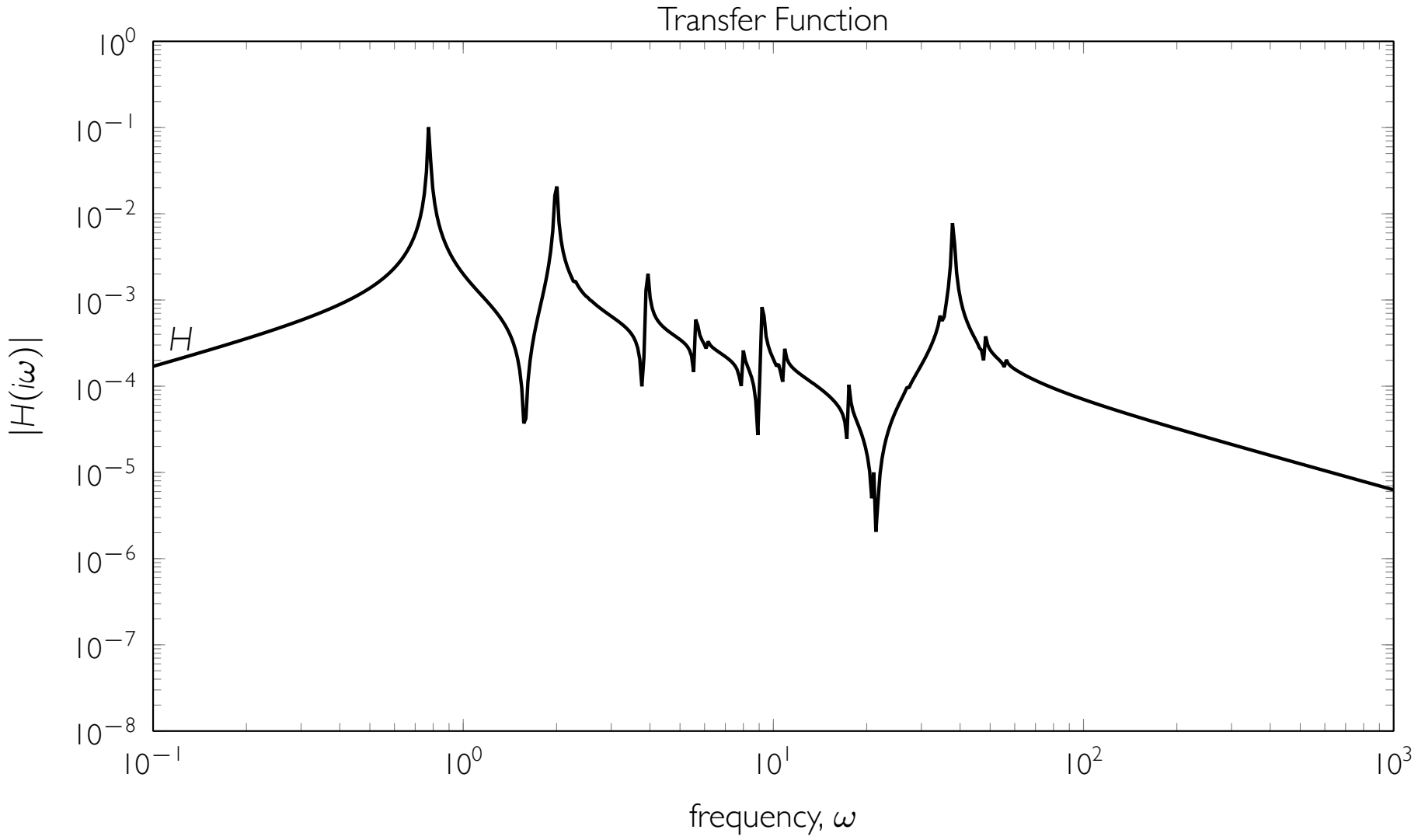
- Then it must be the case for all poles of H_r^ℓ
- This implies local optimality

Example:
ISS 1R Component (1st input/1st output)



NASA

ISS 1R Component (1st input / 1st output)



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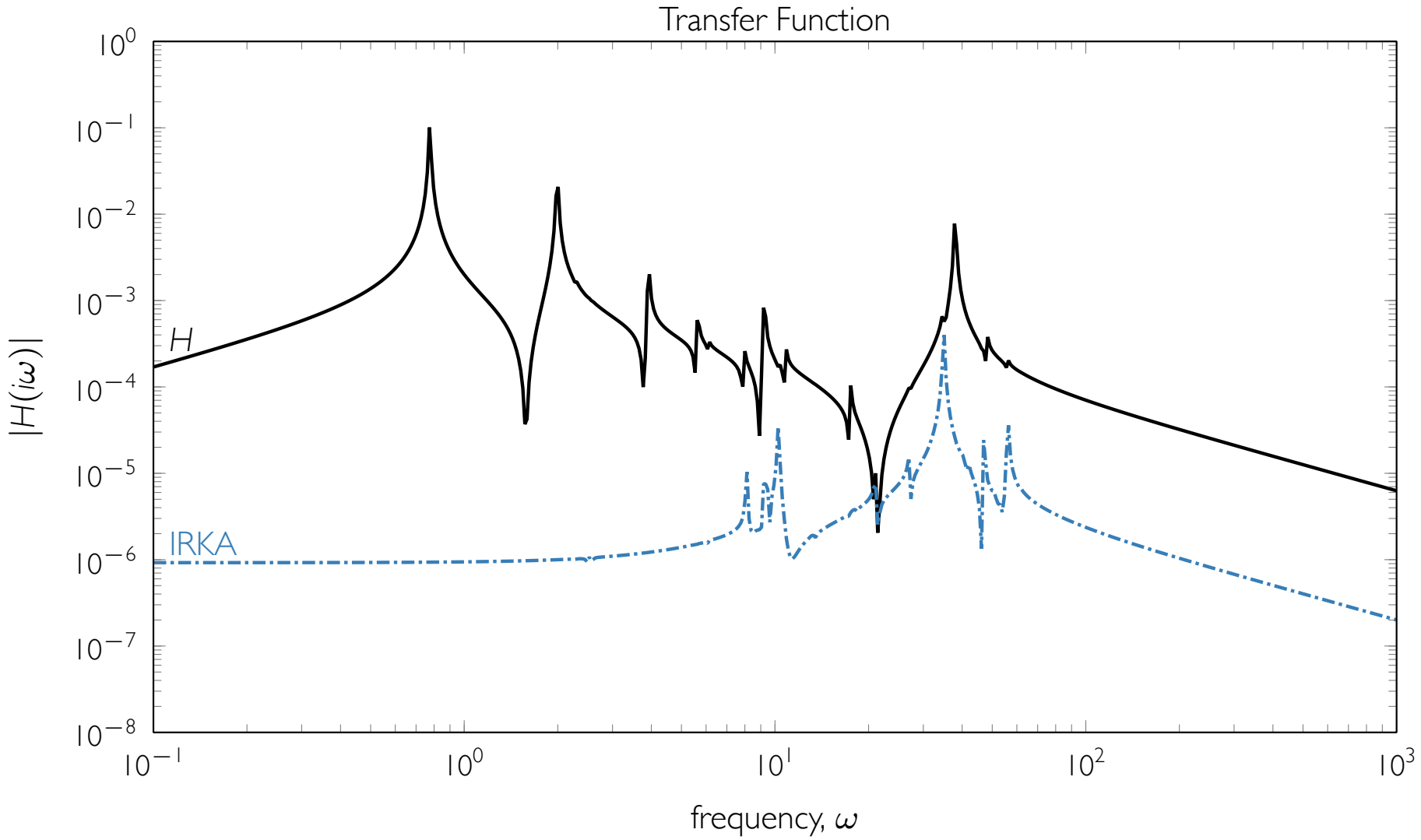

Relative $\mathcal{H}_2$ residual ($r = 28$): **IRKA**: 0.020612

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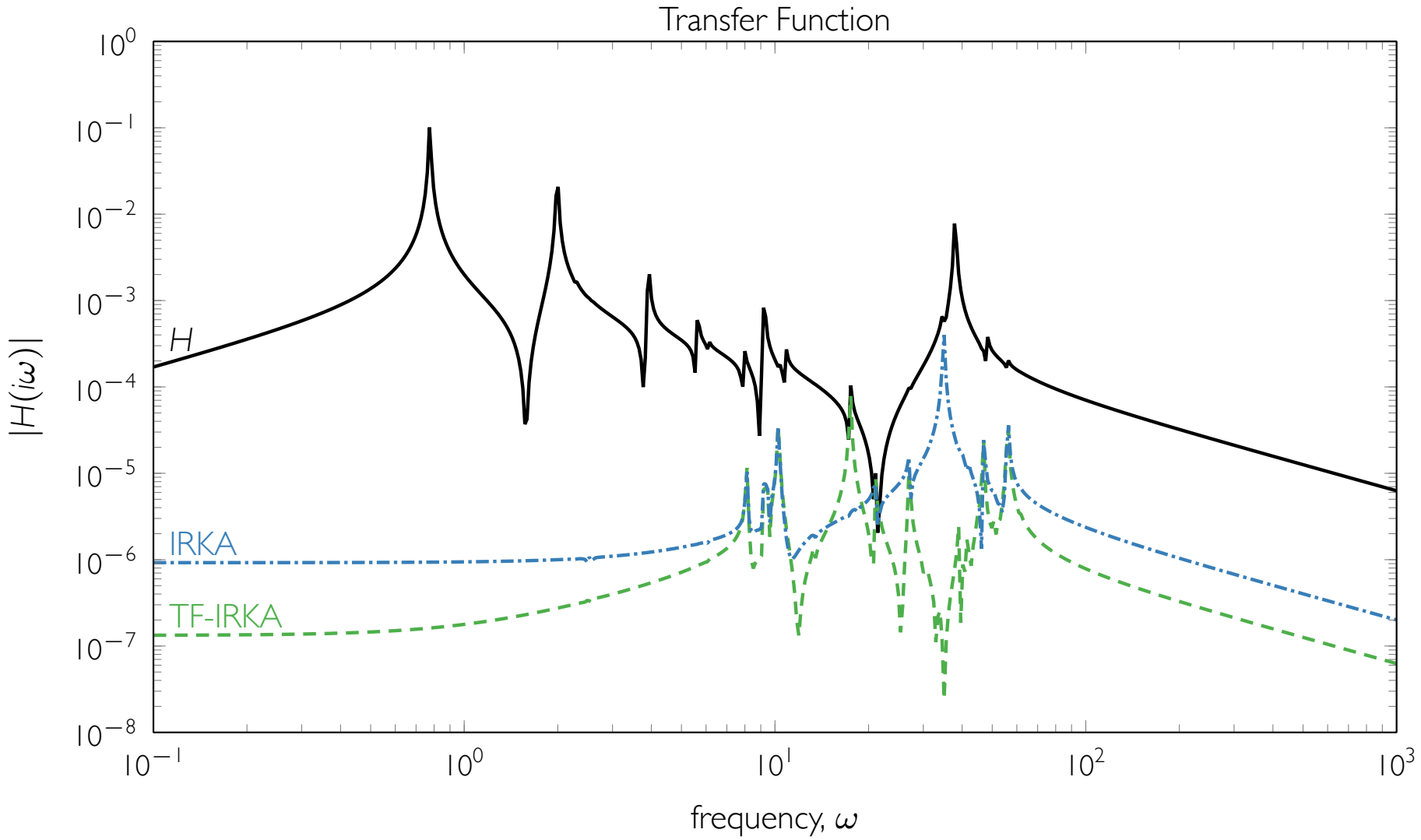

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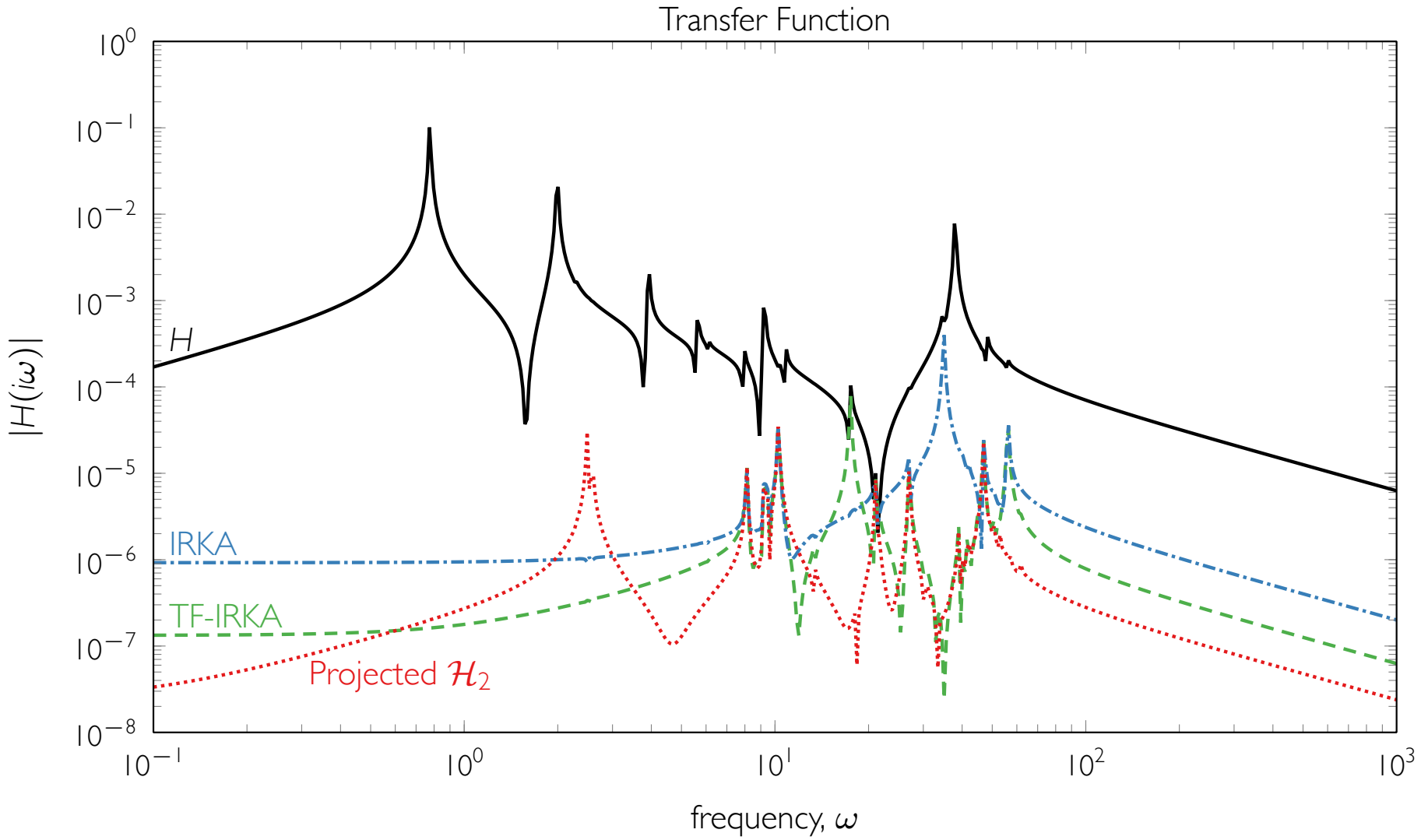

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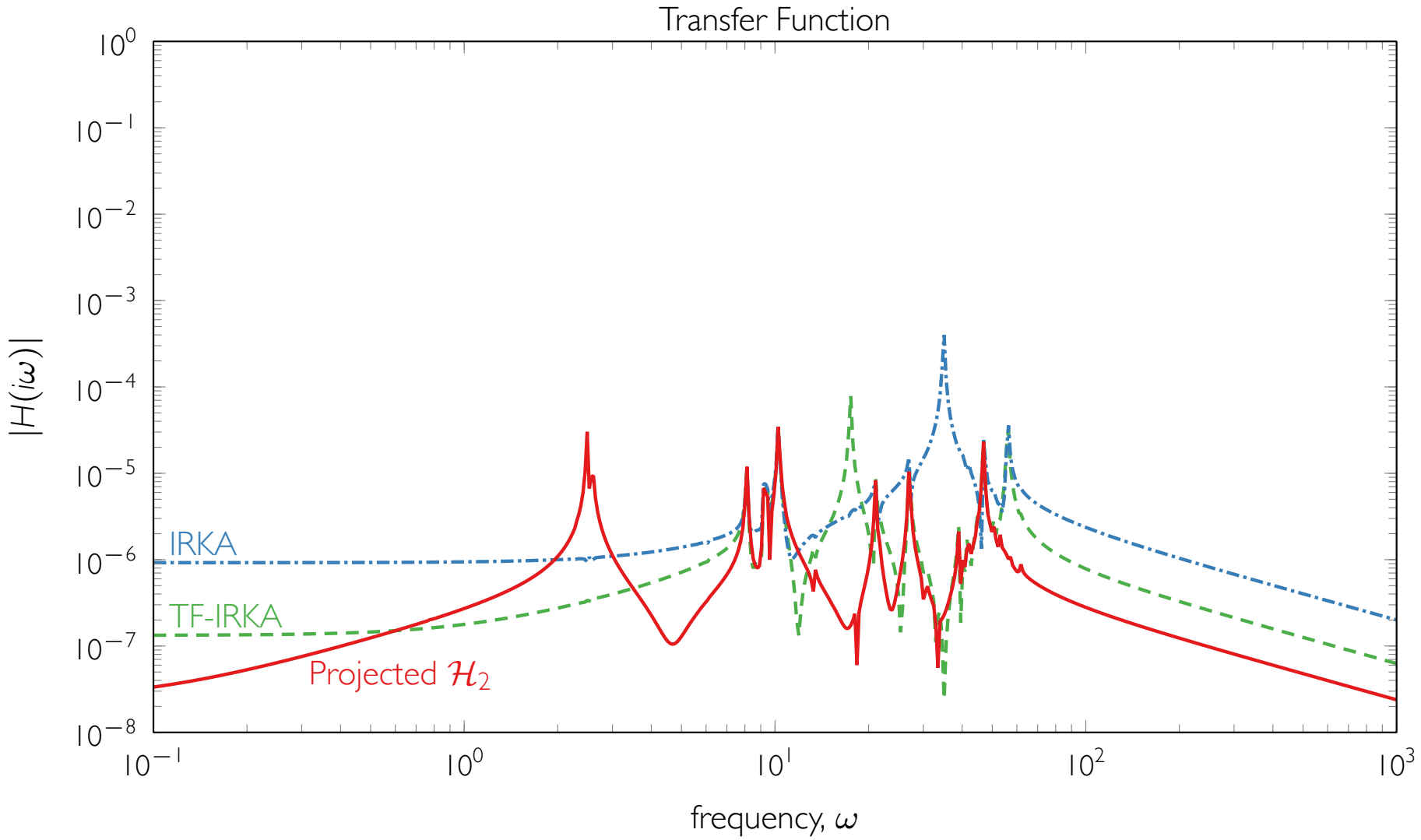

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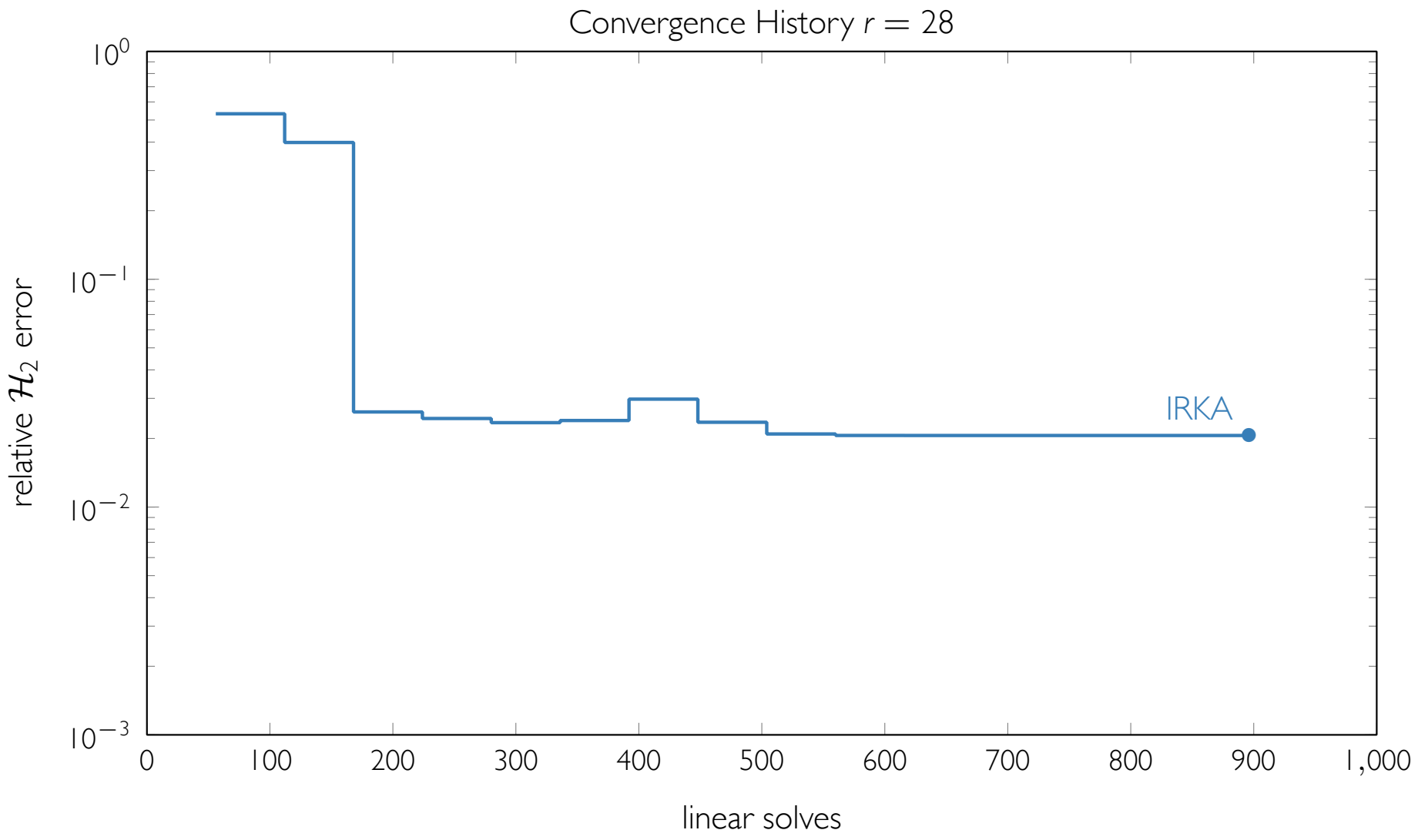

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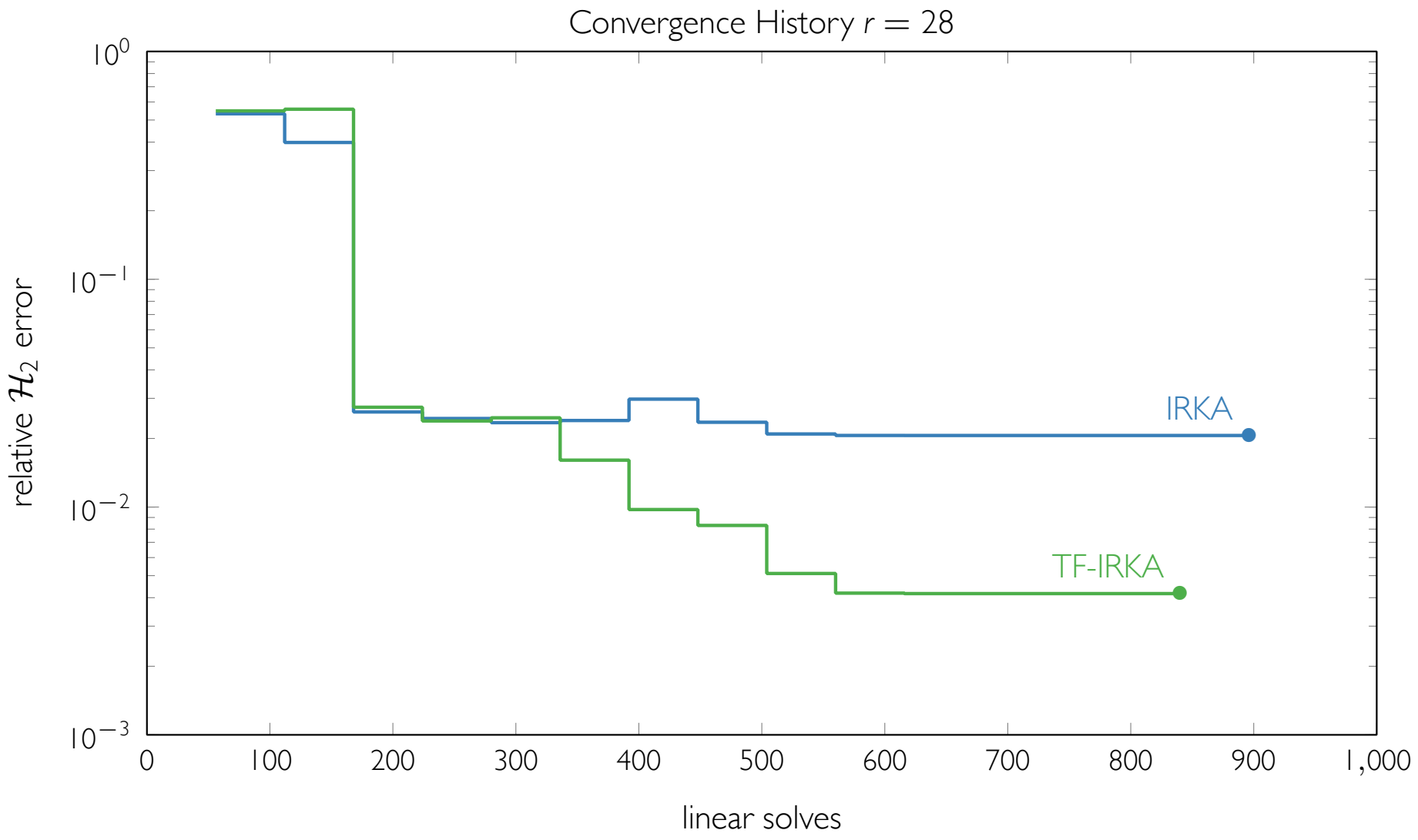

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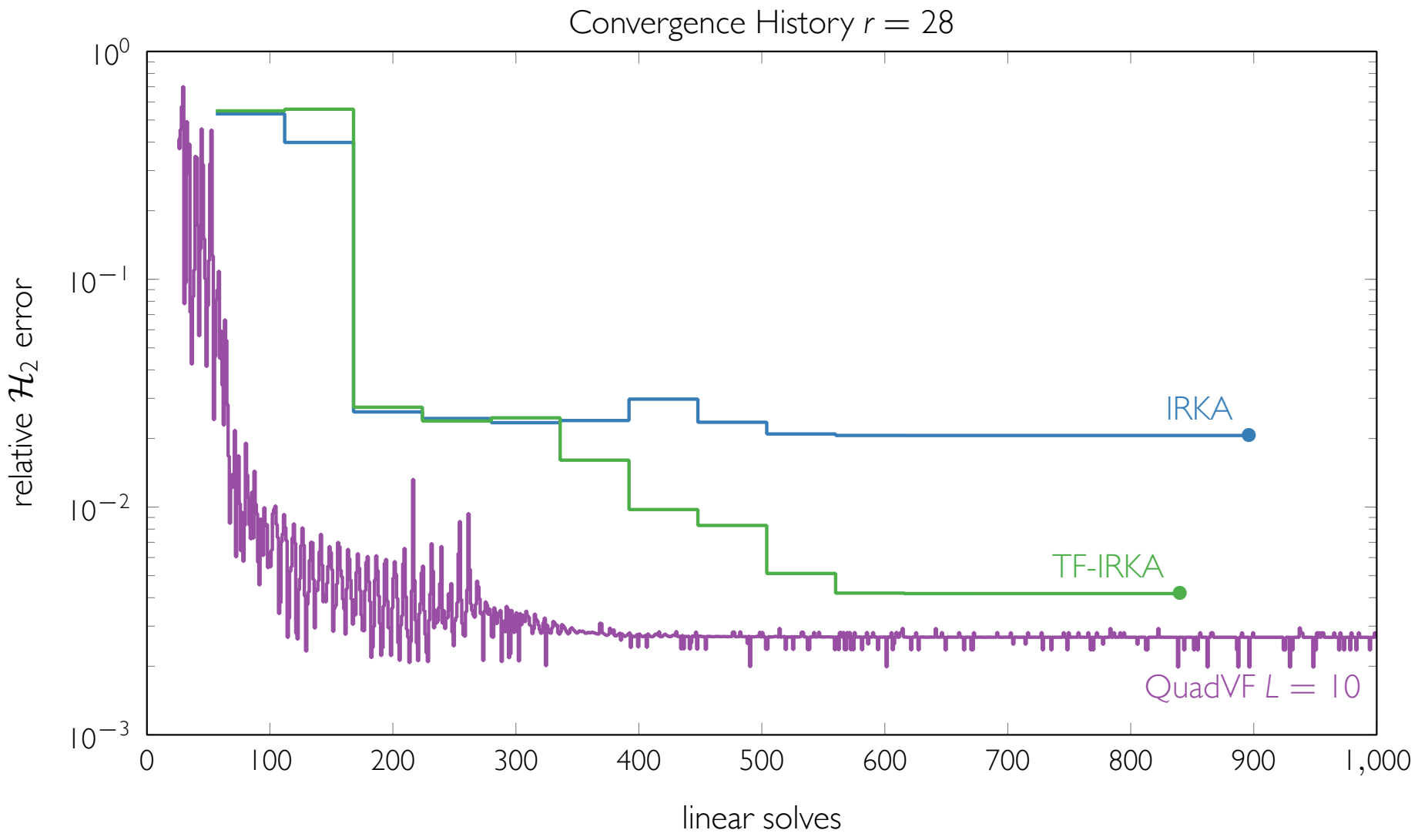

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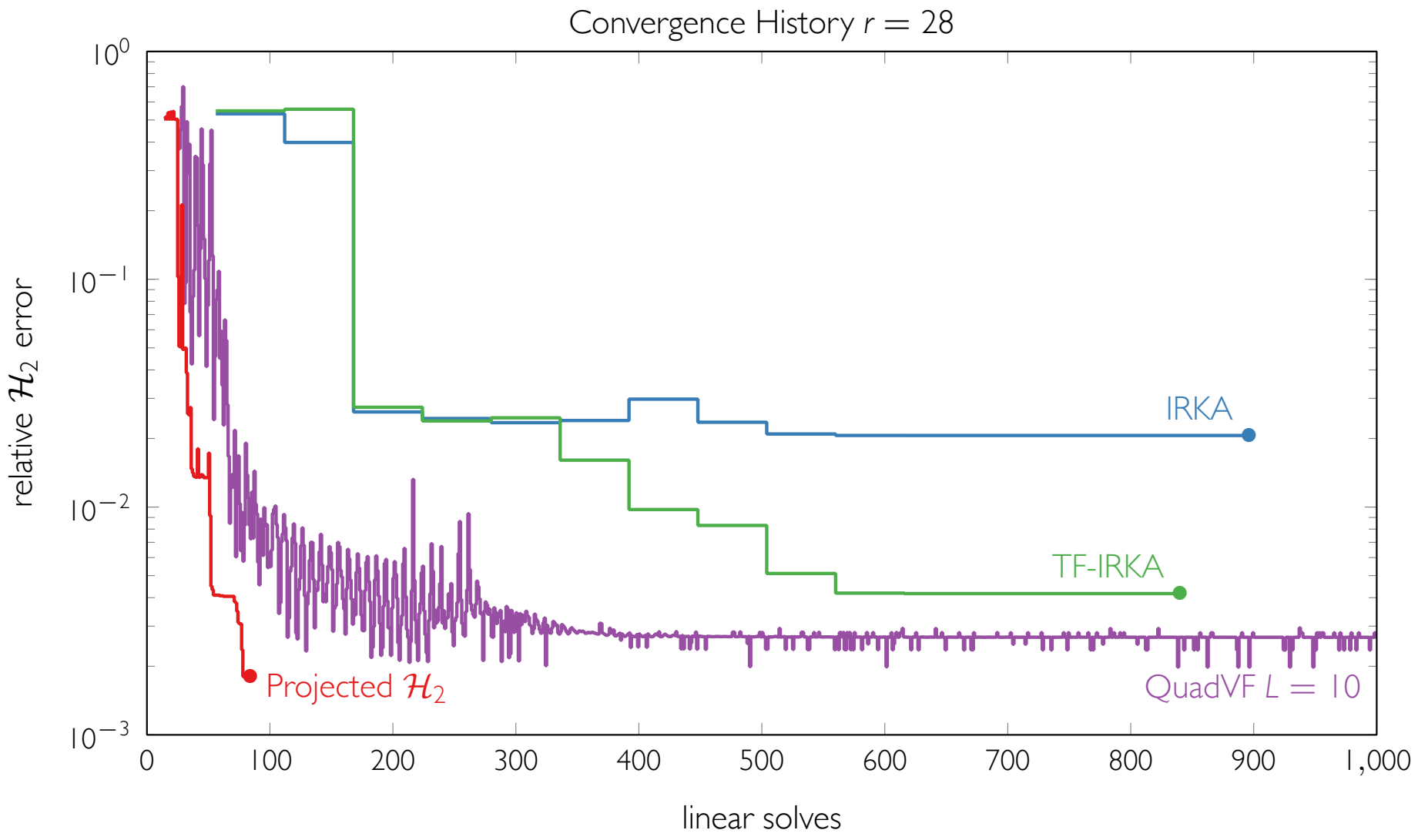

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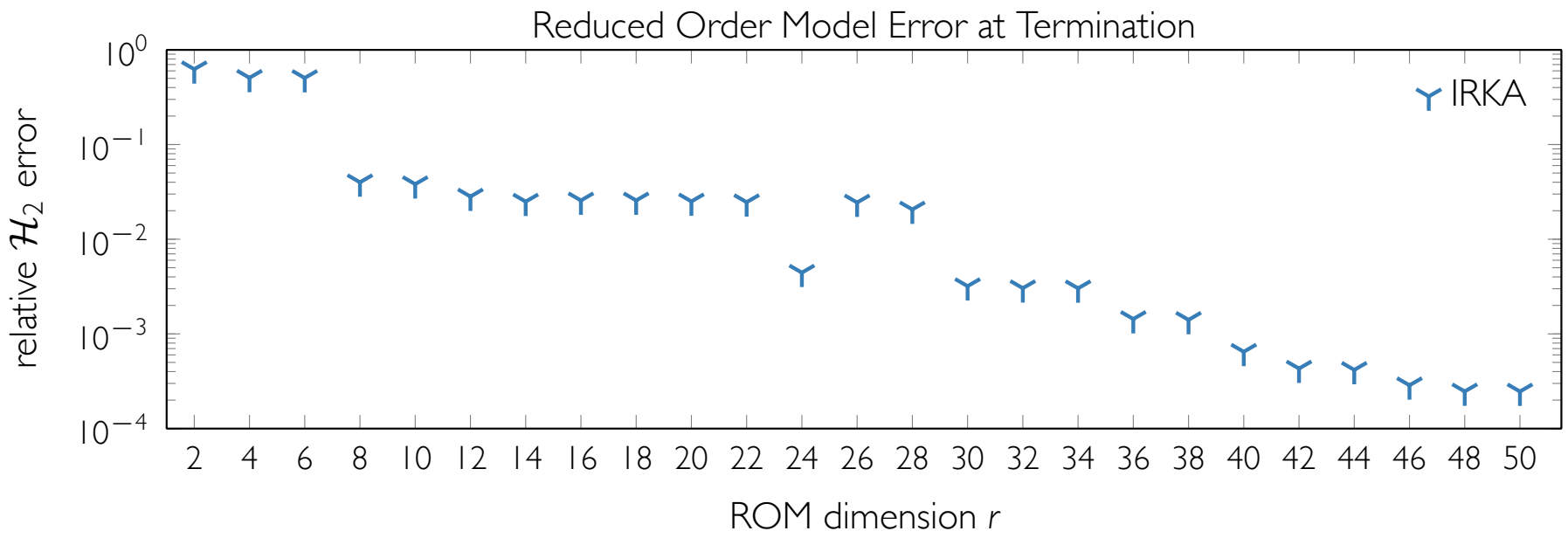

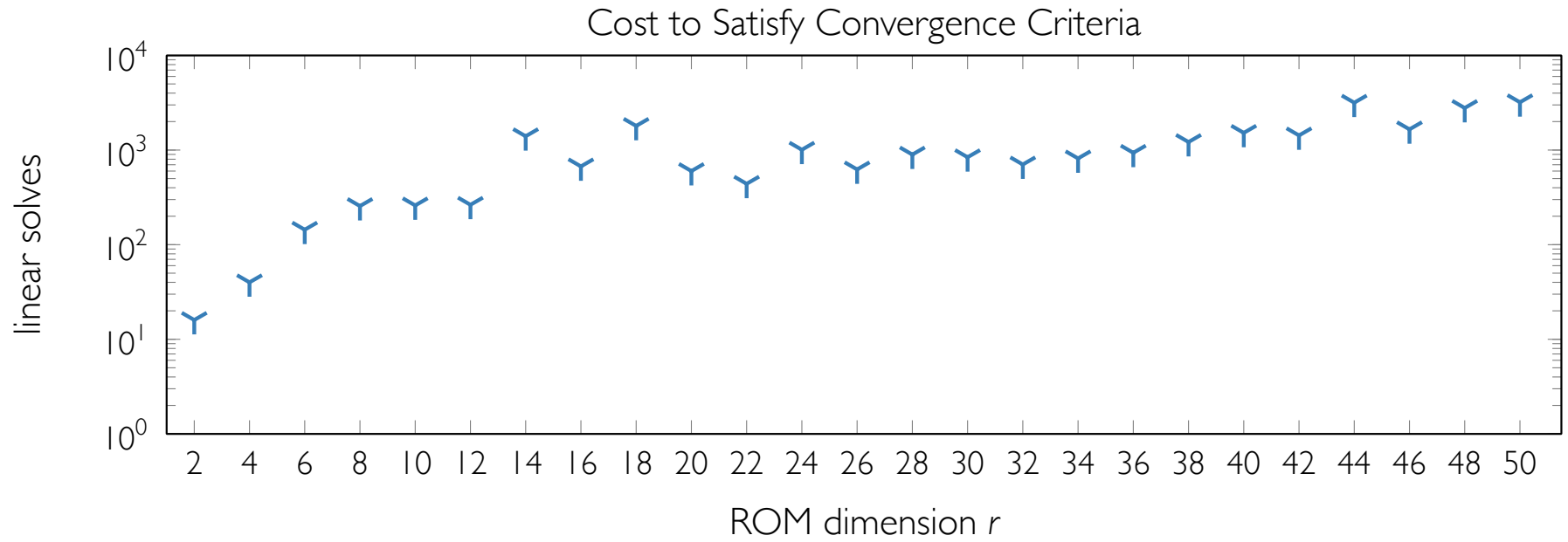

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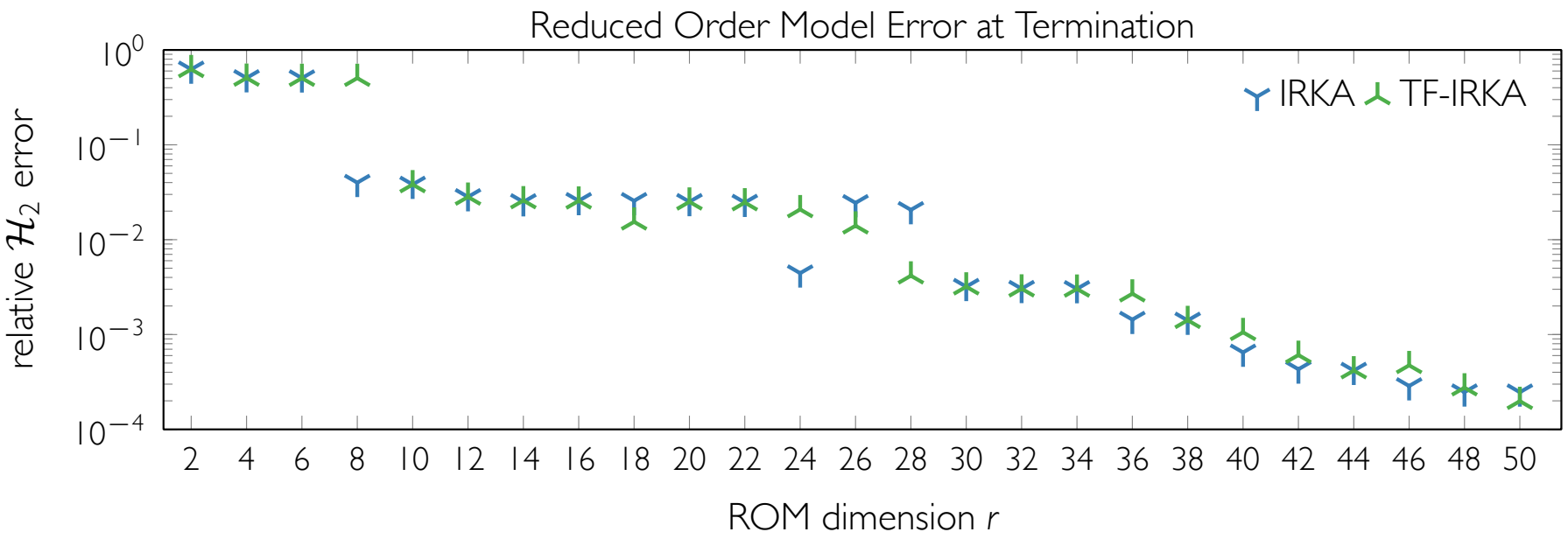

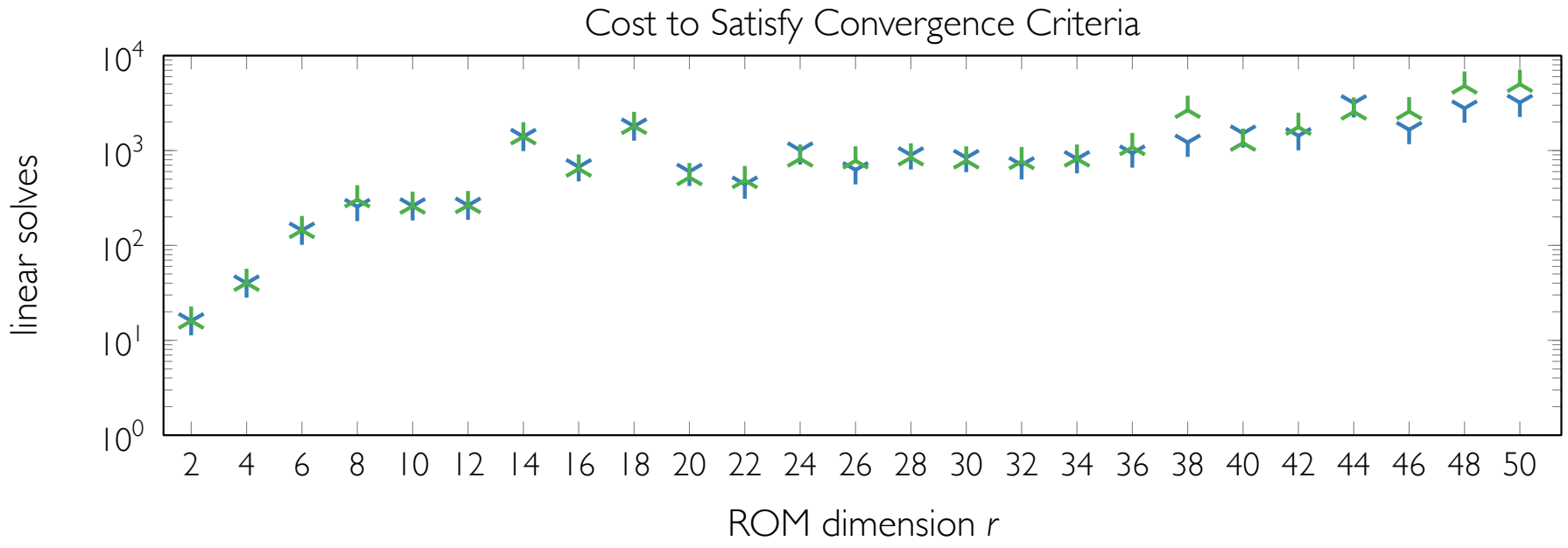

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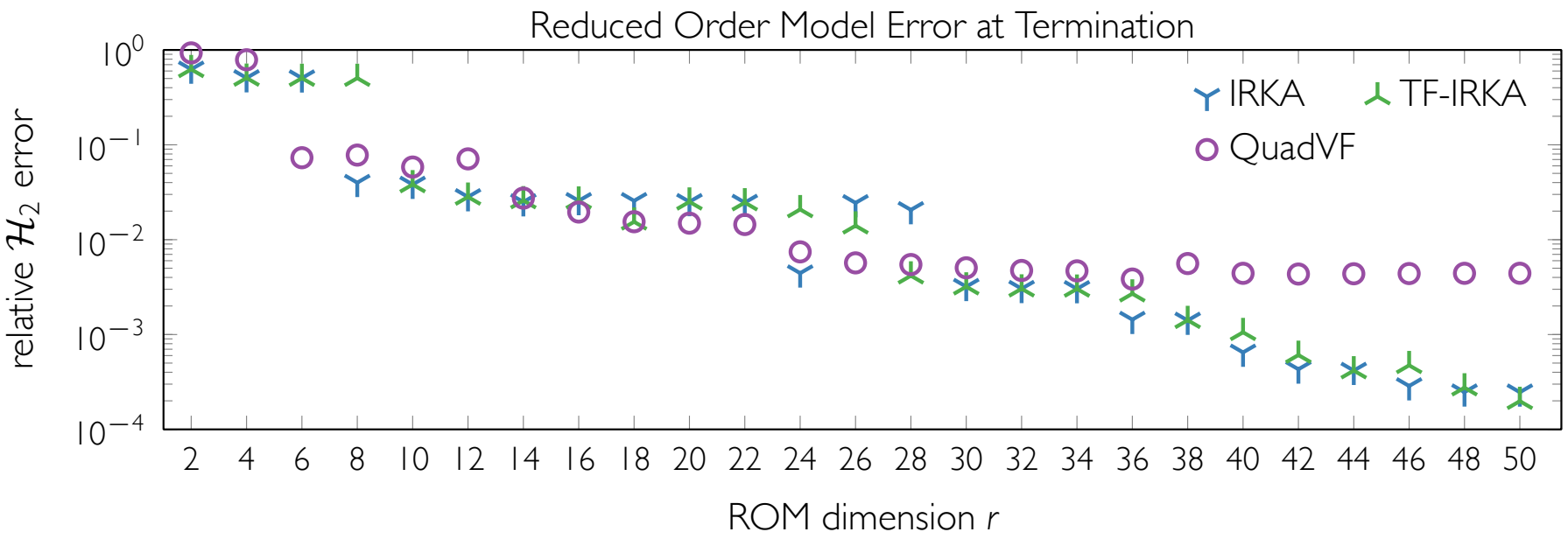

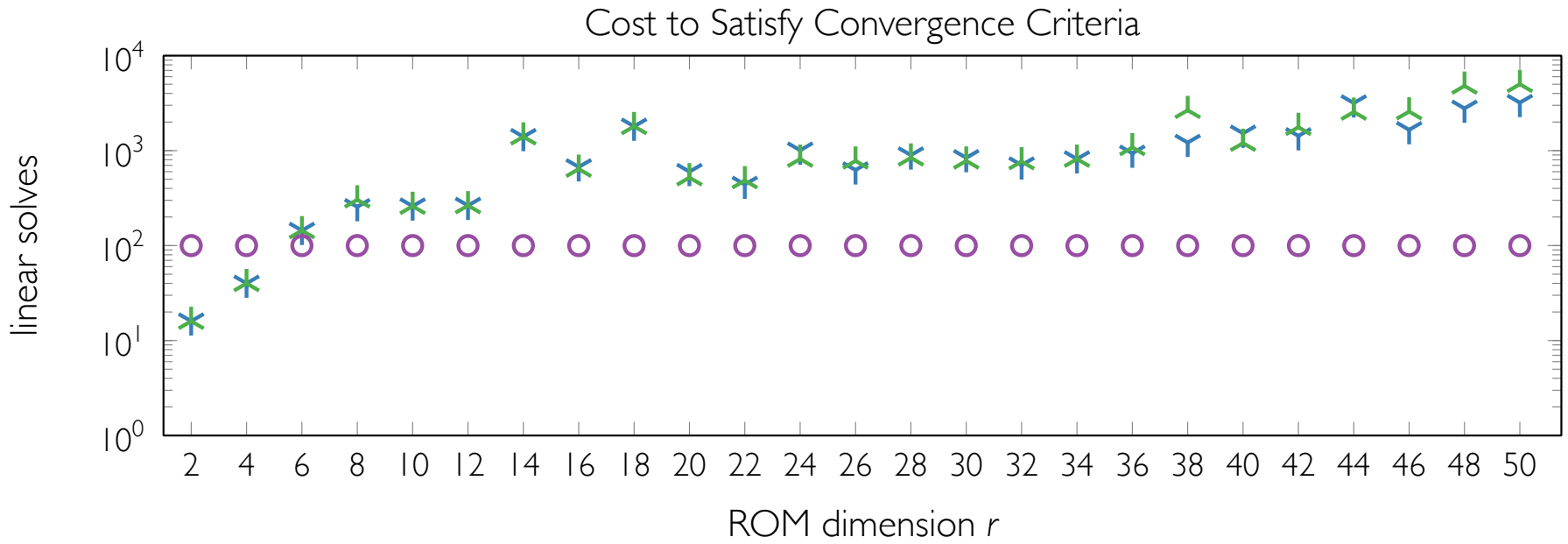

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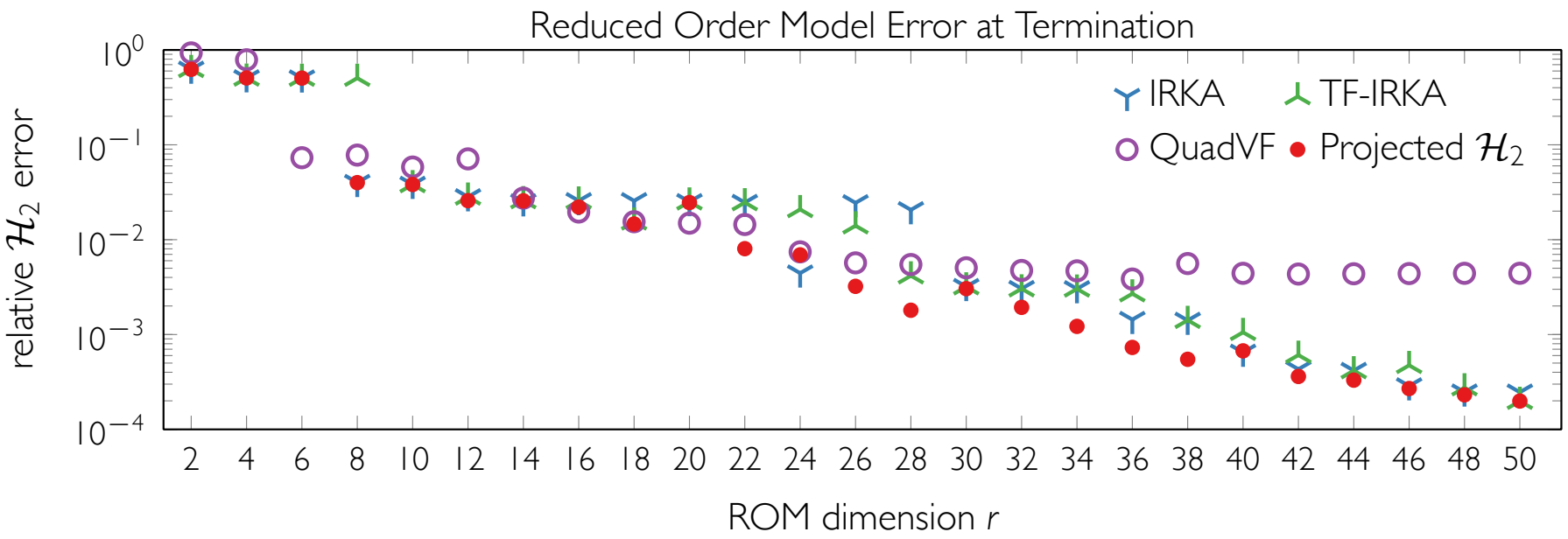
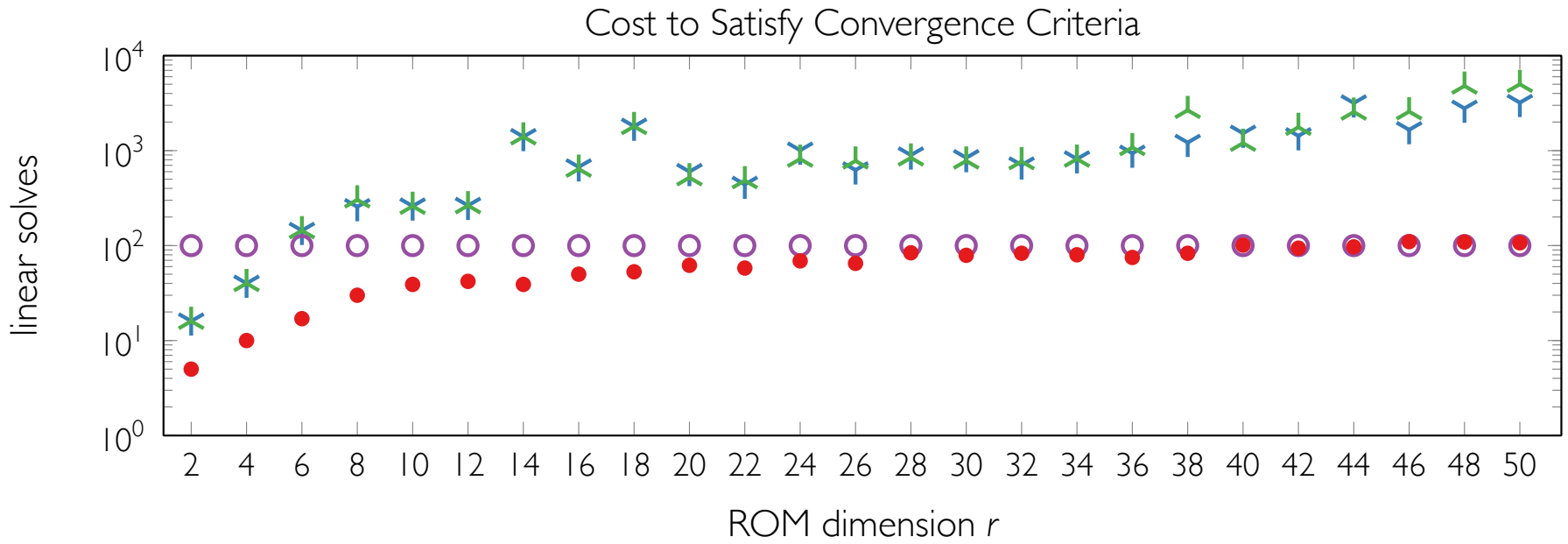




Example: Delay Differential Equation

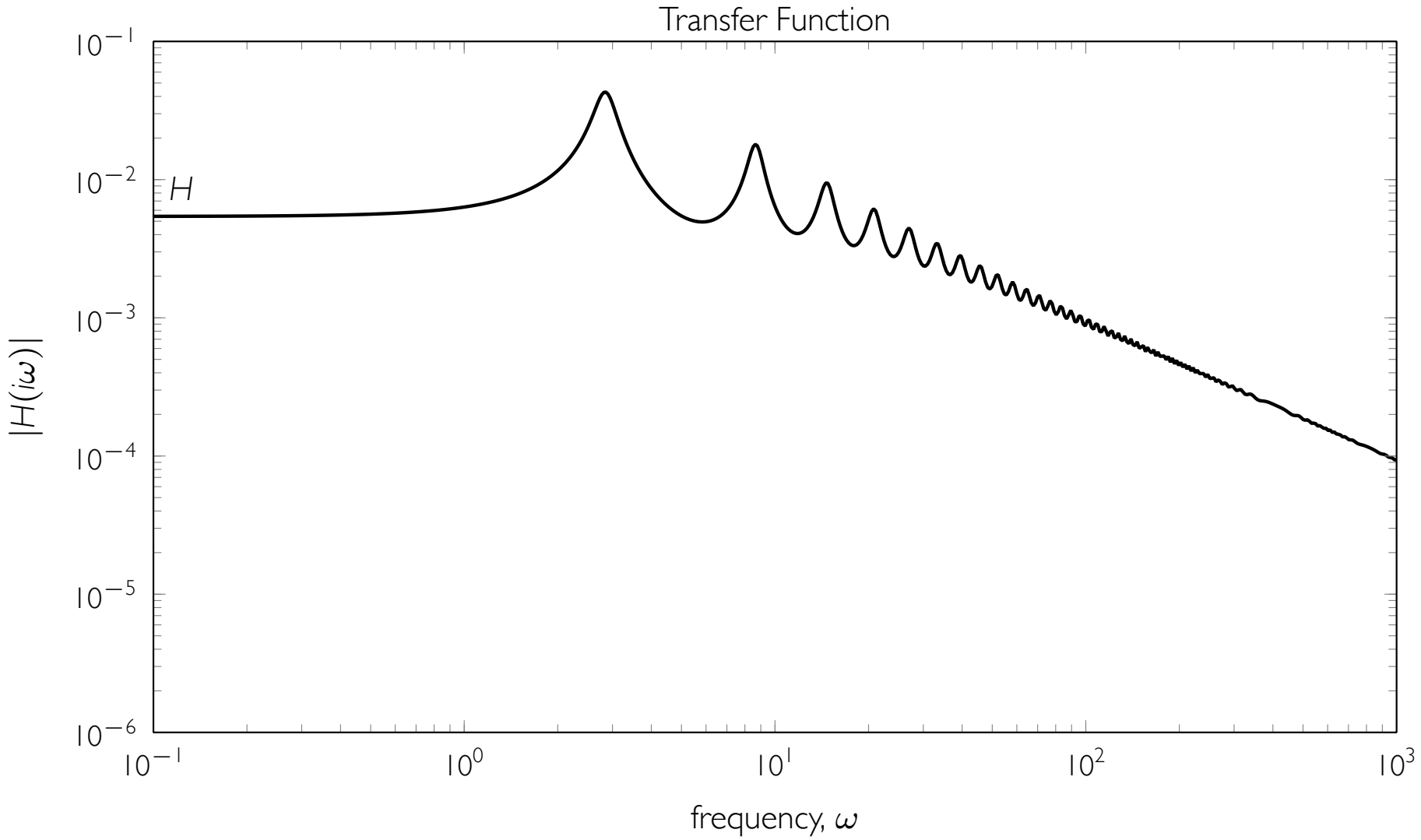
$$H(z) = \mathbf{c}^\top (z\mathbf{E} - \mathbf{A}_0 - e^{-\tau z}\mathbf{A}_1)^{-1}\mathbf{b} \qquad \text{where}$$

$$\mathbf{E} := \frac{2}{\sqrt{\epsilon}}\mathbf{I} + \mathbf{T}, \quad \mathbf{A}_0 := \frac{2+2\rho}{\tau\rho}\left(\mathbf{T} - \frac{2}{\sqrt{\epsilon}}\mathbf{I}\right), \quad \mathbf{A}_1 := \frac{2-2\rho}{\tau\rho}\left(\mathbf{T} - \frac{2}{\sqrt{\epsilon}}\mathbf{I}\right),$$

$\mathbf{T}$ ones on superdiagonal, subdiagonal, and the $(1,1)$ and (n,n) entries
$n = 1000$, $\tau = 1$, $\rho = 0.1$, $\epsilon = 0.01$, $\mathbf{b} = \mathbf{c}$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$

[Beattie & Gugercin '09]

Delay System



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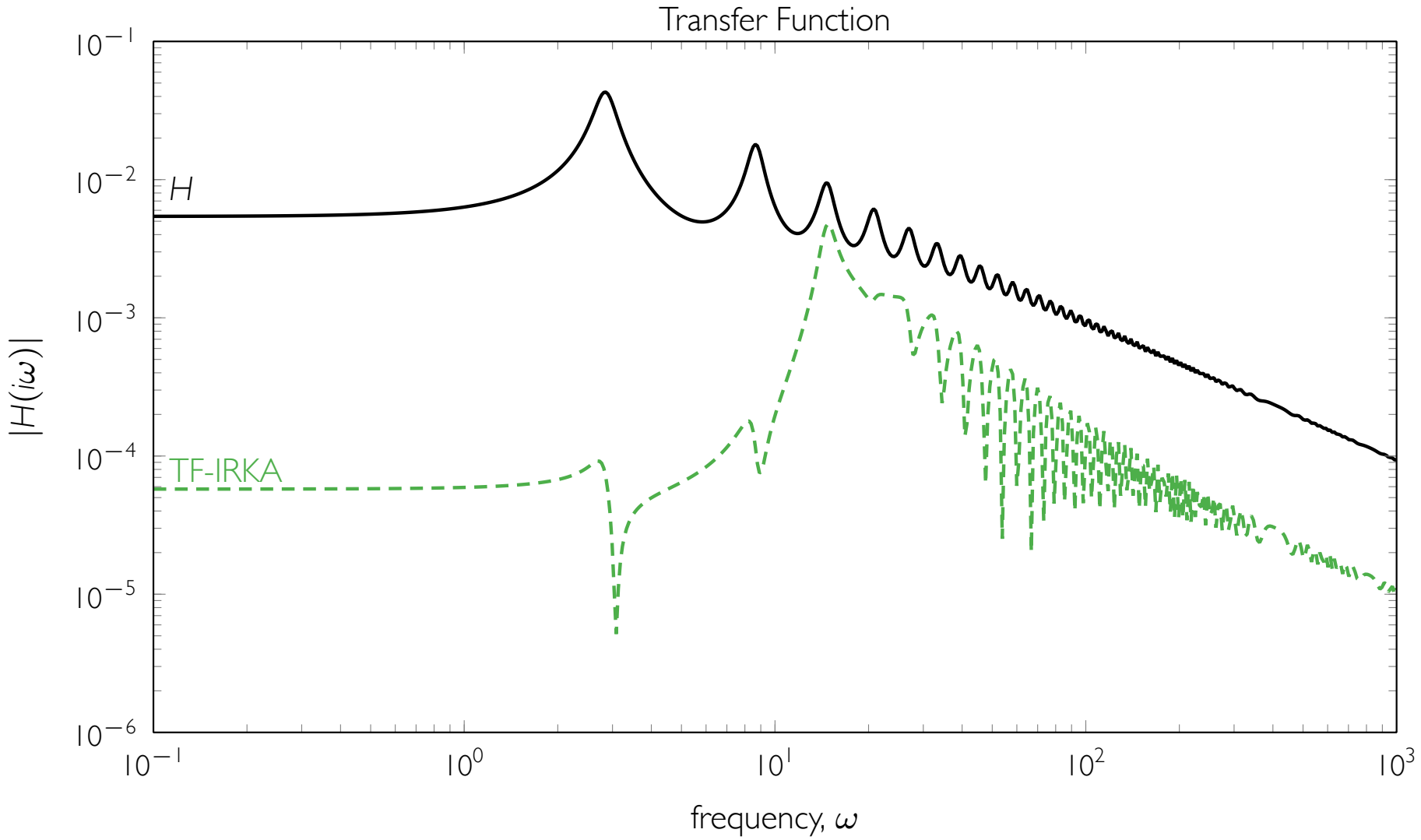

Relative $\mathcal{H}_2$ residual ($r = 6$): **TF-IRKA**: 0.1818

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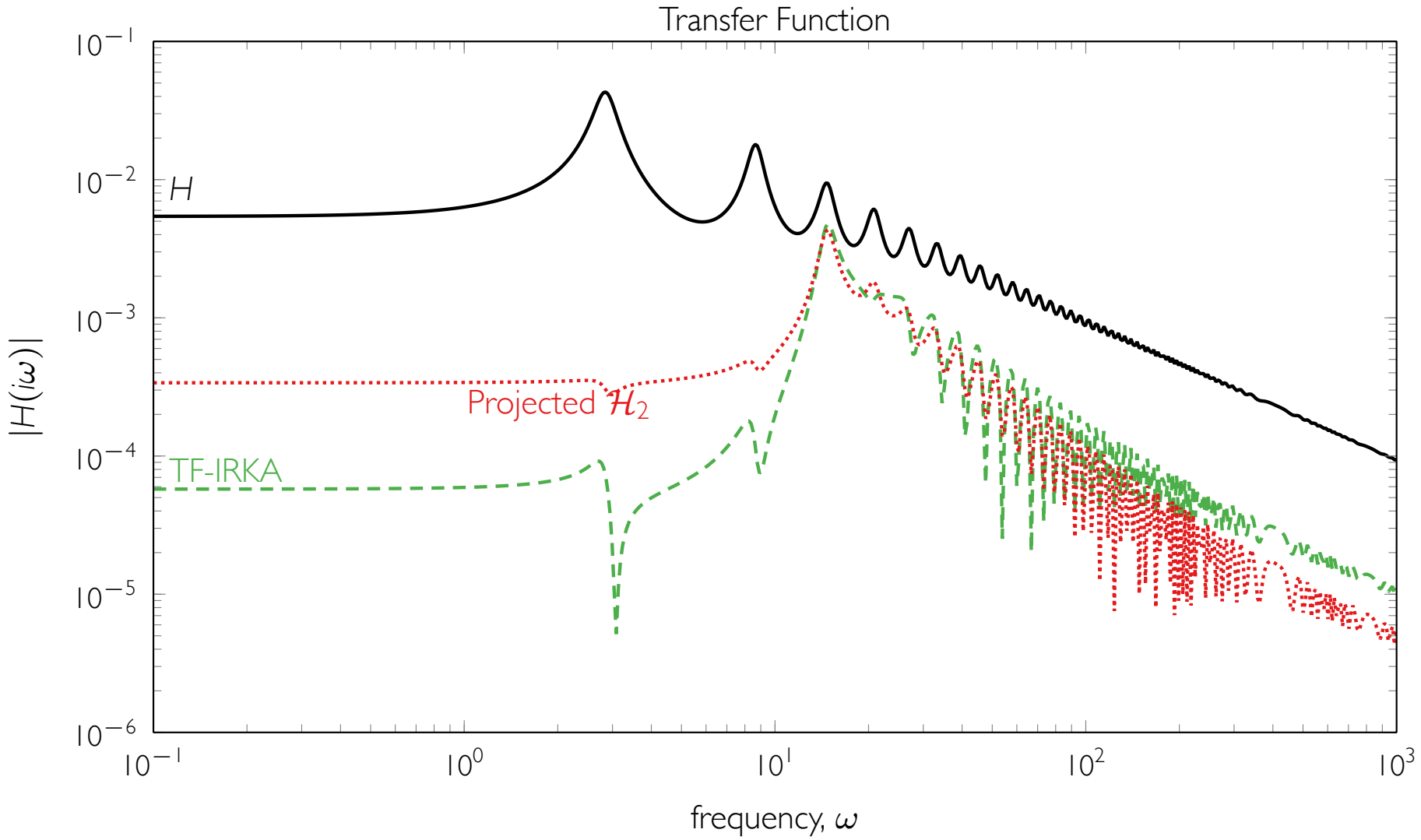

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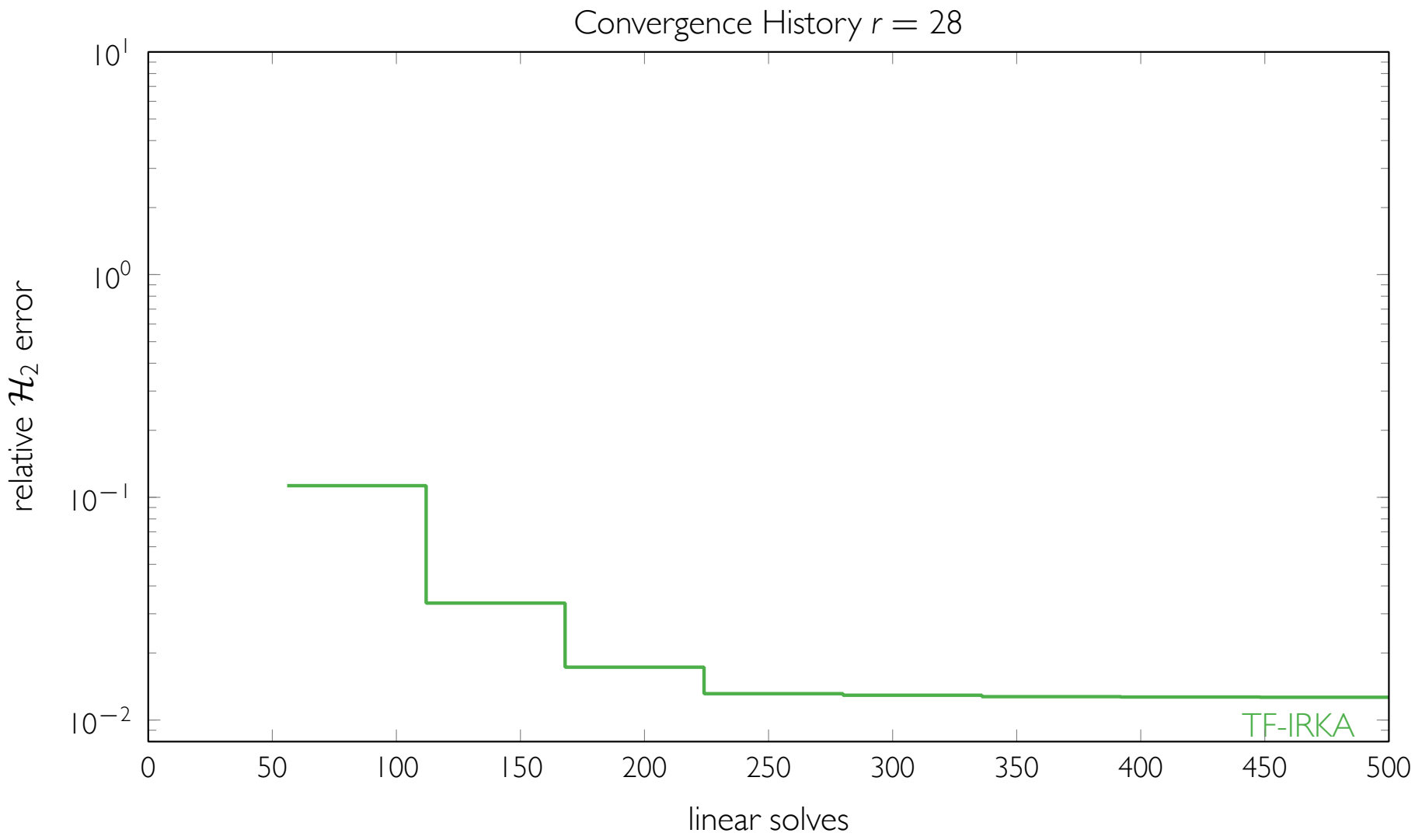

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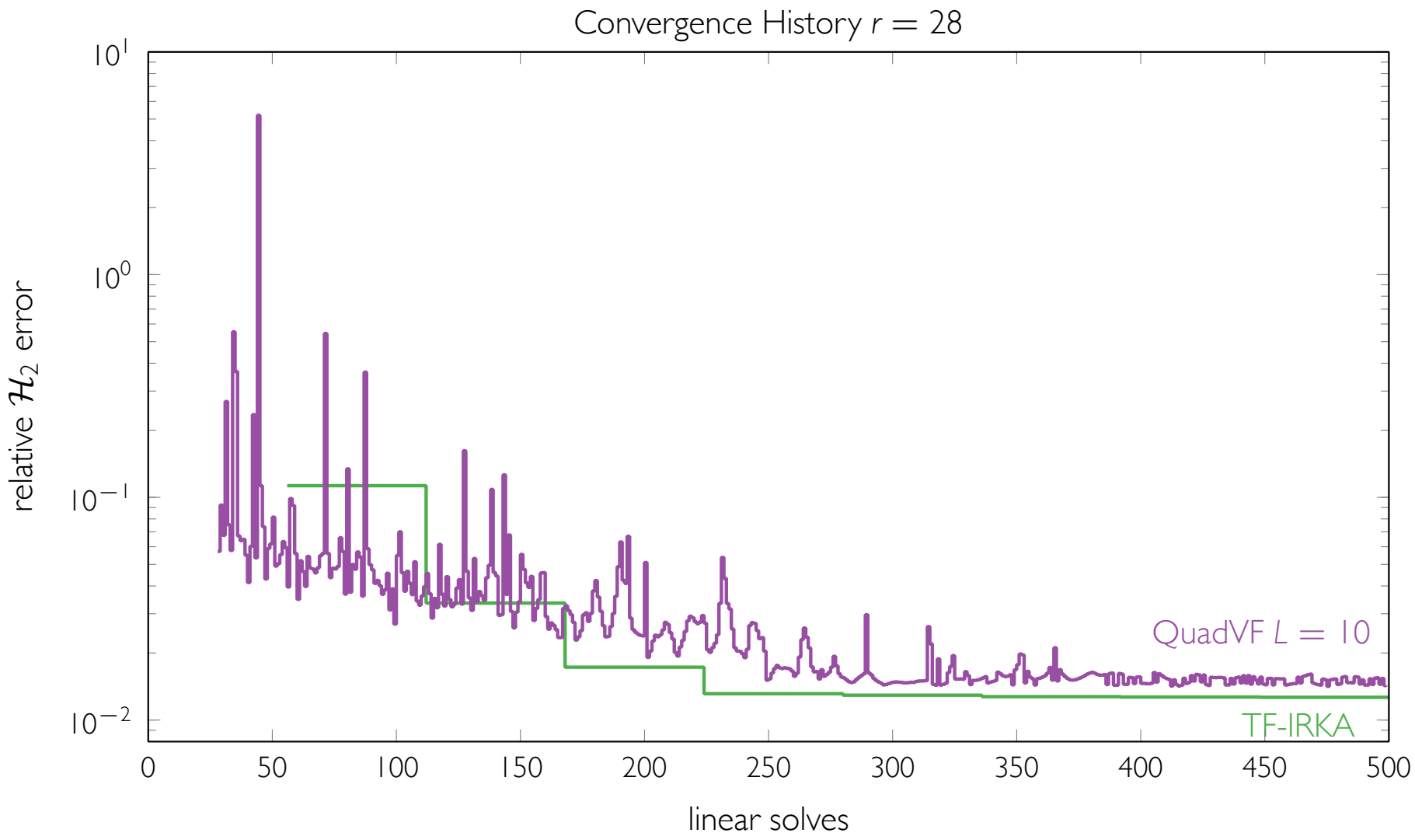

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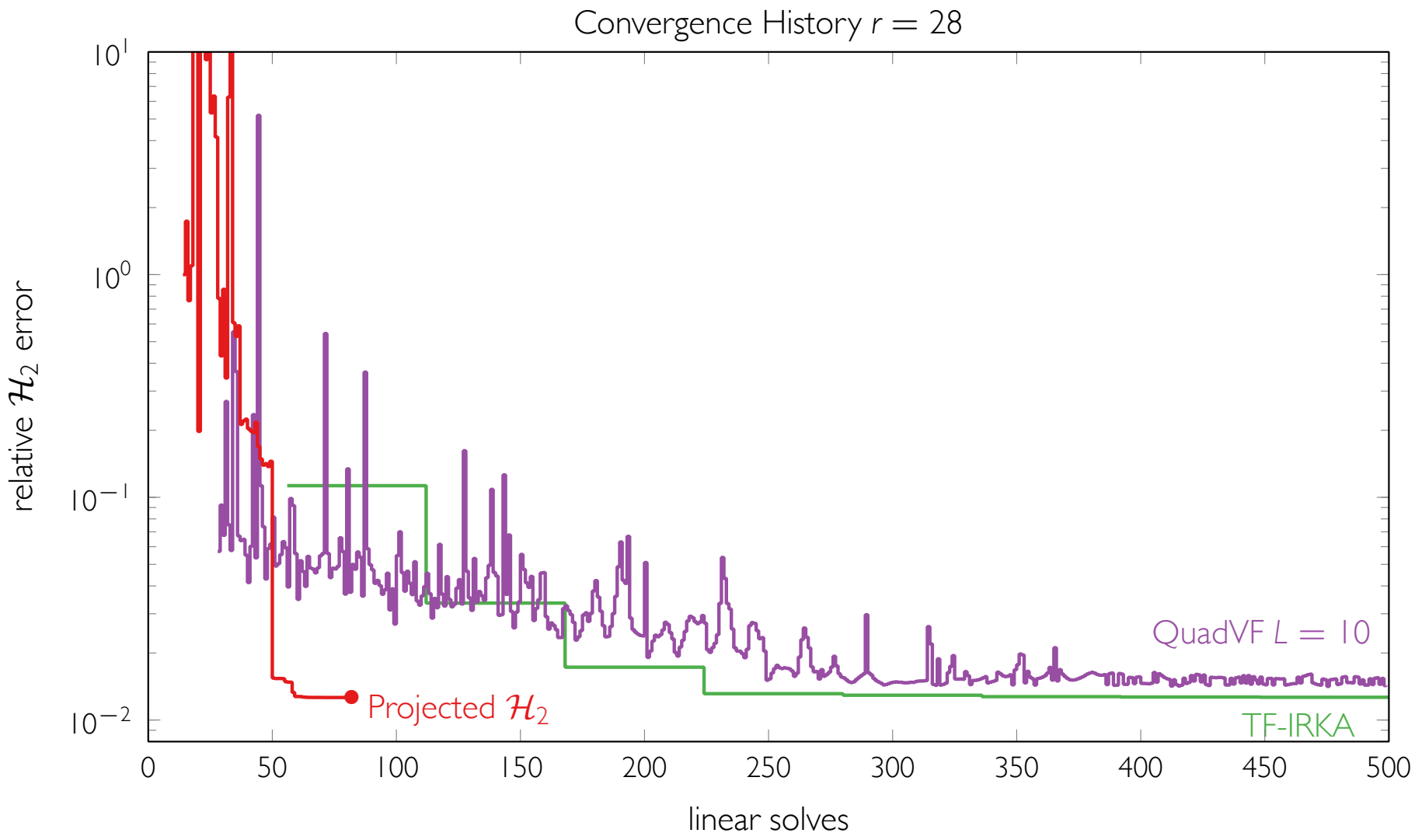

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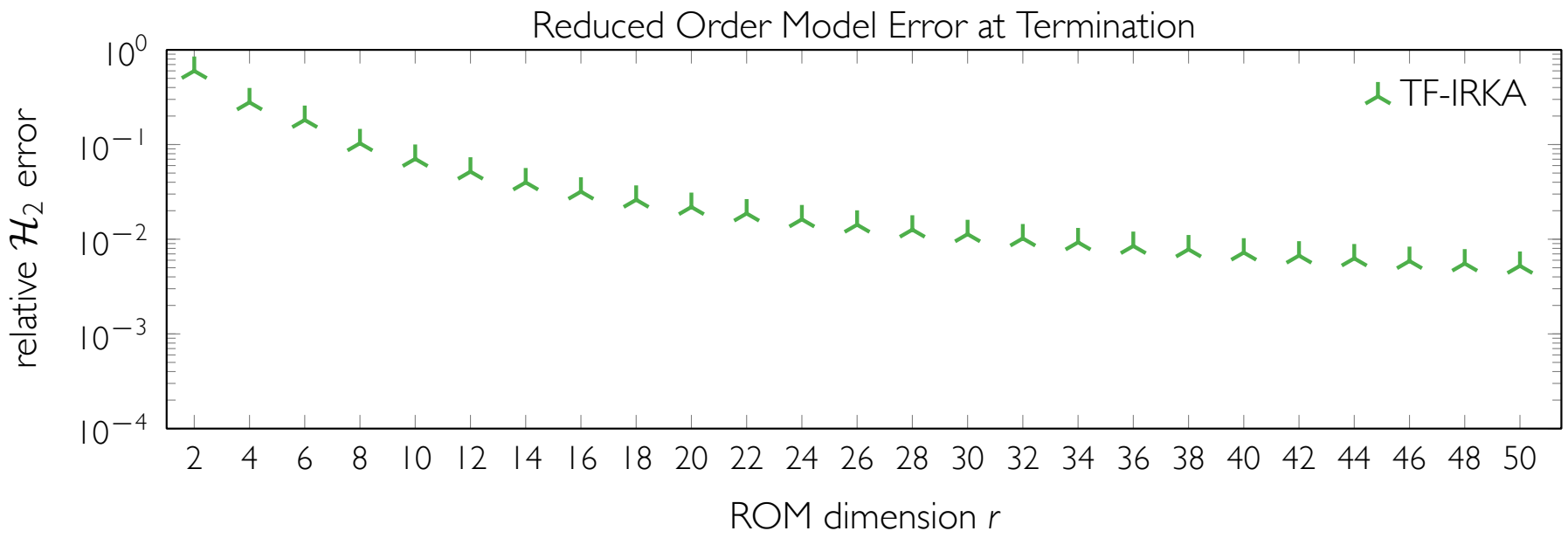

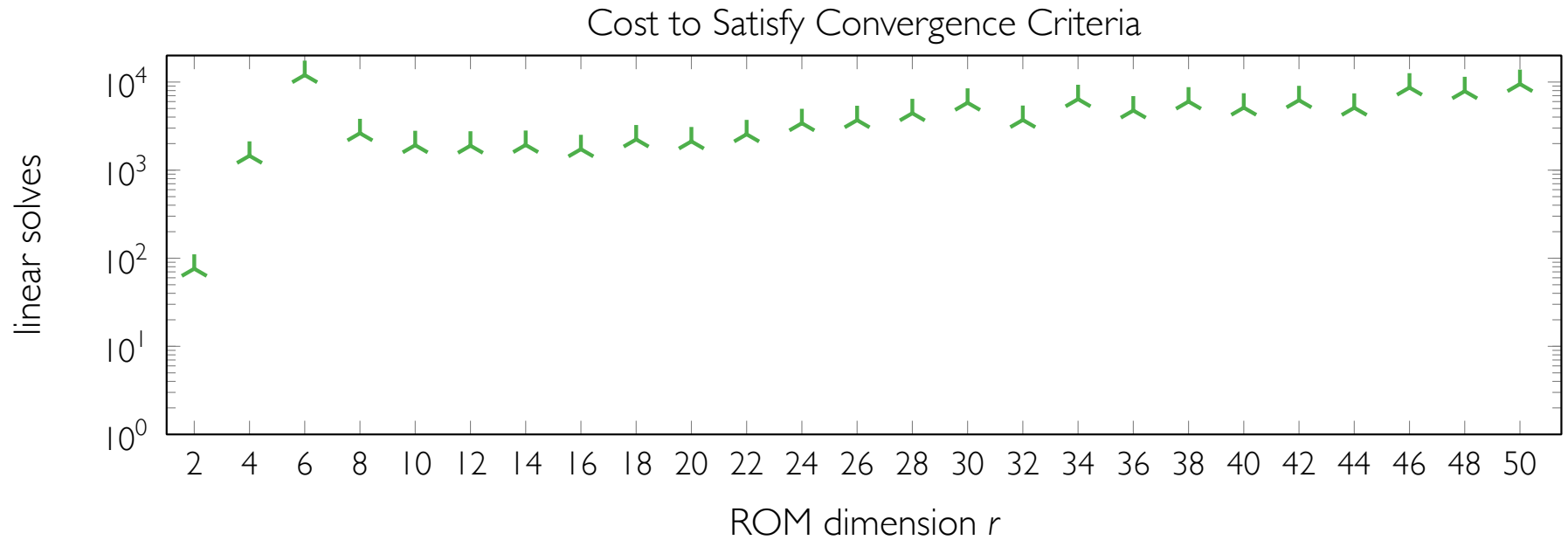

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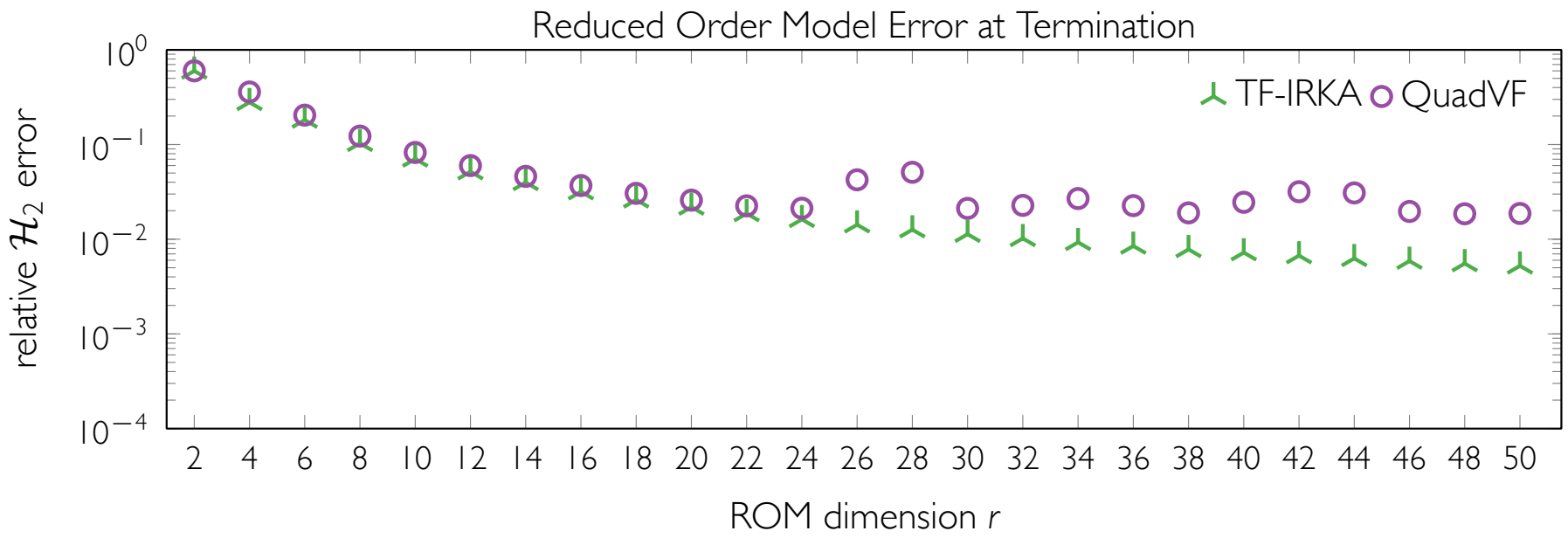

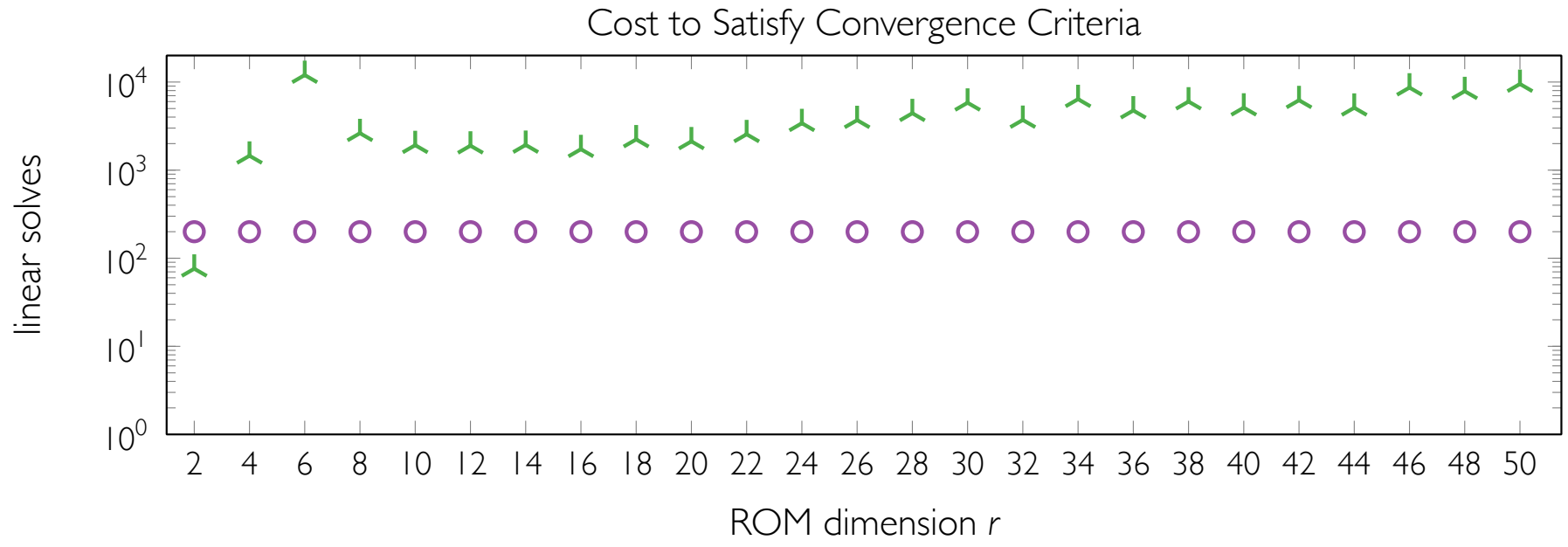

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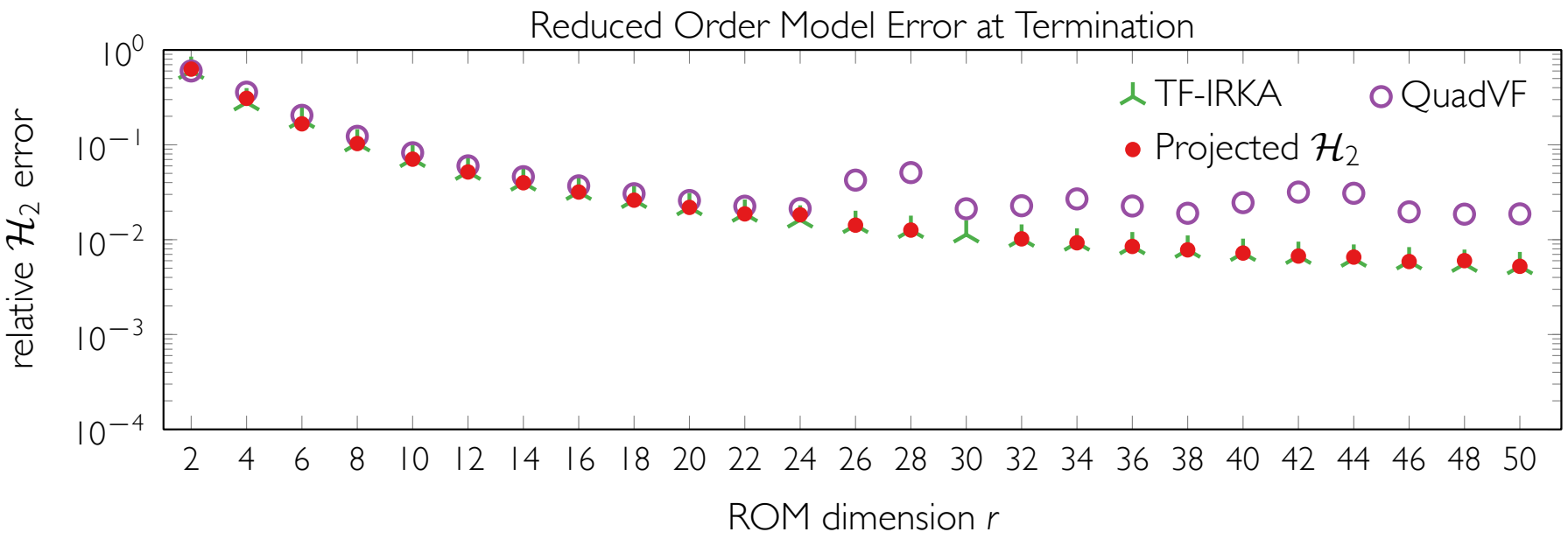

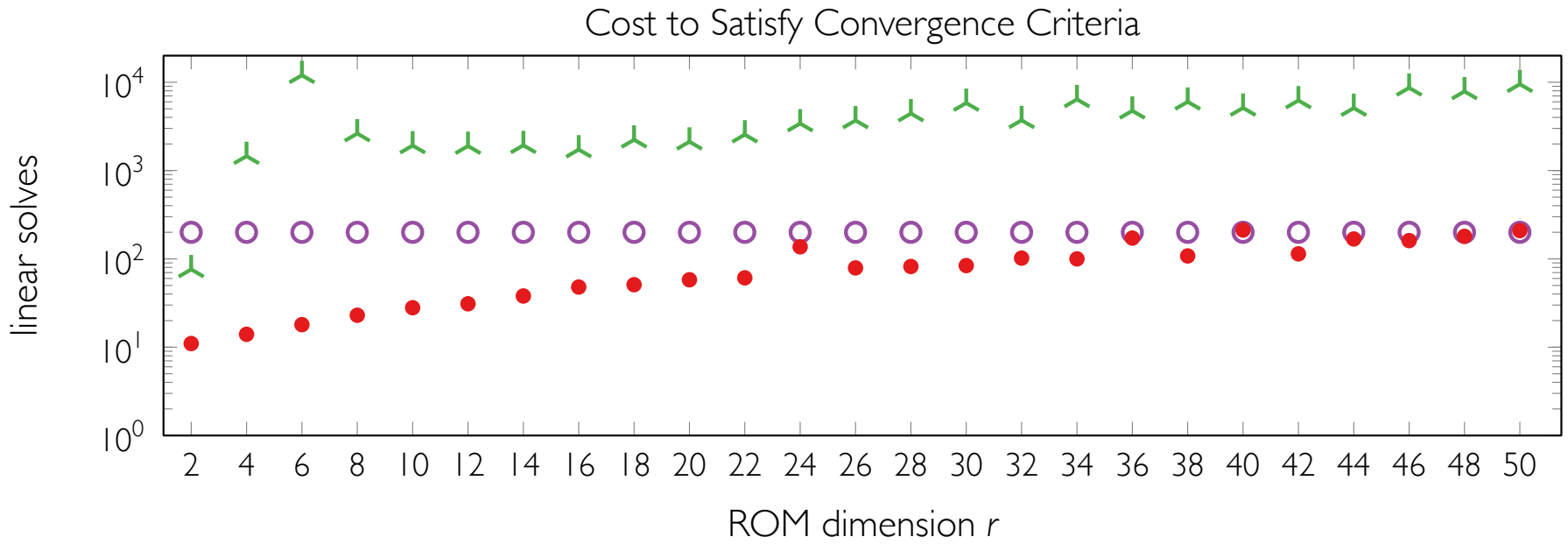

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Structured State-Space ROMs

Structured ROMs

i.e., we assume $\mathbf{A}$ has structure $\mathbf{A} = \sum_j \alpha_j \mathbf{A}_j$

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i.e., $\mathbf{x}''(t) = \mathbf{A}_1 \mathbf{x}'(t) + \mathbf{A}_0 \mathbf{x}(t) + \mathbf{b} u(t)$

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Port-Hamiltonian Structure

with Chris Beattie

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To apply the projected $\mathcal{H}_2$ framework:

- Efficient parameterization & solver
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All are subsets of $\mathcal{S}_r(\mathbb{R})$

$$\mathcal{S}_r^{\mathrm{S}}(\mathbb{R}) \subset \mathcal{S}_r \quad \mathcal{S}_r^{\mathrm{HO}}(\mathbb{R}) \subset \mathcal{S}_r \quad \mathcal{S}_r^{\mathrm{PH}}(\mathbb{R}) \subset \mathcal{S}_r$$

Tangent cone contained in $T_{H_r}\mathcal{S}_r(\mathbb{R})$

the same subspace update can be used!

Broader Classes of ROMs

Weighted $\mathcal{H}_2$ Model Reduction

with Serkan Gugercin

If $W_o, W_i \in \mathcal{H}_2$ and denoting $(F \odot G)(z) = F(z) \cdot G(z)$,

$$\min_{H_r \in \mathcal{S}_r(\mathbb{R})} \|W_o \odot [H - H_r] \odot W_i\|_{\mathcal{H}_2}$$

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typically W_o, W_i proper rational, but high order

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Approximation by Delay Systems

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elements of tangent cone have countably infinite poles

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elements of tangent cone have countably infinite poles

In both cases, construct $\mathcal{P}(\boldsymbol{\mu})$ to **approximate** tangent cone (using model reduction?)

Why use our Projected $\mathcal{H}_2$?

Advantages

- **Non-intrusive**: only access $H(\mu)$
- **Fewer evaluations** ($\approx 4 - 10\times$)
- Often **similar or better** error
- Possible extensions
  - structured ROMs
  - weighted norm
  - delay ROMs
  - quadratic bilinear ROMs(?)

Disadvantages

- Each step wall-clock expensive
- Non-trivial to implement
  but code avalible